TOPIC : MEAN VALUE THEOREMS

SUBJECT: MATHEMATICS

SEMESTER: 2, GENERIC ELECTIVE: 2

UNIT: 1

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(91)

Then the function $\phi(x)$ is derivable in $[a,b]$.

Also, $\phi'(a) = f'(a) - k$

$\phi'(a) = f'(a) - k > 0$ & $\phi'(b) = f'(b) - k < 0$.

So, by Darboux theorem, $\exists$ a point $c \in (a,b)$ such that

$$\phi'(c) = 0$$

i.e $f'(c) - k = 0$ i.e., $f'(c) = k$.

~~Rolle's Theorem Suppose $f: [a,b] \to \mathbb{R}$ is a function such that (i) $f(x)$ is continuous in the closed~~

(statement)

Rolle's Theorem Suppose $f(x)$ is a function defined over the closed interval $[a,b]$ is such that

(i) $f(x)$ is continuous over the closed interval $a \le x \le b$

(ii) $f'(x)$ exists in the open interval $a < x < b$

and (iii) $f(a) = f(b)$.

Then there is a point $\xi \in (a,b)$ such that

$$f'(\xi) = 0.$$

Proof Since $f(x)$ is continuous over the closed interval $[a,b]$ it is bounded there and attains its bounds. In other words if

$M = \sup\{f(x) : a \le x \le b\}$ & $m = \inf\{f(x) : a \le x \le b\}$

then there exist points $\xi, \eta \in [a,b]$ such that $f(\xi) = M$ and $f(\eta) = m$.

Now, if $M = m$, then $f(x) = M \;\; \forall x \in [a,b]$ i.e., $f(x)$ is a constant throughout the interval $[a,b]$ and in that case $f'(x) = 0 \;\; \forall x \in (a,b)$.

On the other hand if $M > m$ then as $f(a) = f(b)$, either M or m is distinct from $f(a)$. We suppose that $M \neq f(a)$. Then $M = f(\xi) > f(a)$ and $M = f(\xi) > f(b)$. Hence $\xi \neq a$ & $\xi \neq b$. So, $\xi \in (a,b)$. By the given condition $f'(\xi)$ exists.

We assert that $f'(\xi) = 0$. Because if $f'(\xi) > 0$ then there would exist a $\delta_1 > 0$ such that if $x \in (\xi, \xi + \delta_1)$, then $f(\xi) < f(x)$, which is not possible and if $f'(\xi) < 0$, then there would exist a $\delta_2 > 0$ such that if $x \in (\xi - \delta_2, \xi)$, $f(\xi) < f(x)$, which is not possible. Hence we must have $f'(\xi) = 0$.

If $m \neq f(a)$, we can show in a similar way that $\eta \in (a,b)$ and $f'(\eta) = 0$.

Ex:- Let $f(x) = |x|$; $x \in [-1, 1]$.
Then $f(x)$ is continuous in $[-1, 1]$, $f'(x)$ exists at all $x \in [-1, 1]$ except at $x = 0$ & $f(-1) = f(1)$.
There is no point $c \in (-1, 1)$ such that $f'(c) = 0$.

Ex:- Let $f(x) = \frac{1}{x}$ $(0 < x \leq 1)$
$= 1$ at $x = 0$.

Then $f(x)$ is not continuous at $x = 0$. We note that there is no $x \in (0, 1)$ such that $f'(x) = 0$.

Ex:- Let $f(x) = \tan x$, $0 \leq x \leq \pi$.
Then $f(0) = f(\pi) = 0$. $f(x)$ is continuous in $[0, \pi]$ except at $x = \pi/2$. $f'(x) = \sec^2 x$ which doesnot vanish in $[0, \pi)$.

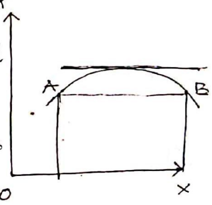
Ex:- Let $f(x) = \frac{1}{x} + \frac{1}{1-x}$, $x \in (0, 1)$.
Then $f(x)$ is continuous in $(0, 1)$ {not in $[0, 1]$}
$f'(x) = \frac{1}{(1-x)^2} - \frac{1}{x^2}$. Yet $f'(\frac{1}{2}) = 0$

Rolle's theorem ensures that there exists atleast one point $c \in (a, b)$ such that $f'(c) = 0$. The following example shows that there may exist more than one point in (a, b) at which f' vanishes.

Ex:- Let $f(x) = \sin x$, $0 \leq x \leq 5\pi$
$f'(x) = \cos x$.

Then $f'(x) = 0$ at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$.

Geometric Interpretition of Rolle's Theorem



If the graph of $y = f(x)$ has the ordinates at two points A and B equal and if the graph be continuous throughout the interval from A to B except possibly at the two extreme points A and B, then there must exist at least one point on the curve intervening between A and B where the tangent is parallel to the x-axis.

Lagrange's Mean Value Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is a function such that
(i) f is continuous in the closed interval $[a, b]$
(ii) f is derivable in the open interval (a, b),
Then there is a point c $(a < c < b)$ such that

$$f(b) - f(a) = (b - a) f'(c).$$

Proof If $f(a) = f(b)$, then the theorem reduces to the Rolle's Theorem. So, we assume that $f(a) \neq f(b)$ and construct a function $\phi : [a,b] \to \mathbb{R}$ as follows :-

$$\phi(x) = f(x) + Ax \quad ; \quad x \in [a,b]$$

where A is a constant to be chosen in such a way that $\phi(a) = \phi(b)$. Thus,

$$f(a) + Aa = f(b) + Ab$$

i.e $\quad A(b-a) = -\{f(b) - f(a)\}$

i.e $\quad A = -\dfrac{f(b)-f(a)}{b-a}$.

As $f(x)$ is continuous in $[a,b]$, derivable in (a,b) and Ax is continuous in $[a,b]$, derivable in (a,b) it follows that $\phi(x)$ is continuous in $[a,b]$, derivable in (a,b). Also $\phi(a) = \phi(b)$. In other words $\phi(x)$ satisfies all the conditions of Rolle's theorem in $[a,b]$. Hence there is a point $c \in (a,b)$ such that

$$\phi'(c) = 0$$

i.e $\quad f'(c) + A = 0$

i.e $\quad f(b) - f(a) = (b-a) f'(c)$.

Other forms of Lagrange's M.V.T.

I). We have
$$f(b) - f(a) = (b-a) f'(c) \quad \text{where } a < c < b.$$
So, $0 < c-a < b-a$ i.e $0 < \dfrac{c-a}{b-a} < 1$. Let $\theta = \dfrac{c-a}{b-a}$. Then $0 < \theta < 1$ and $c = a + \theta(b-a)$.

So, $\quad f(b) - f(a) = (b-a) f'\{a + \theta(b-a)\}$ ---- (1)

II) In (1) if we put $b - a = h$, then it reduces to
$$f(a+h) - f(a) = h f'(a+\theta h) \quad \text{- (2)}$$
$0 < \theta < 1$

In (2) if we put $a = 0$, we obtain
$$f(h) - f(0) = h f'(\theta h), \quad 0 < \theta < 1$$
Now putting $h = x$ in the above relation we obtain
$$f(x) = f(0) + x f'(\theta x) \quad \text{---- (3)}$$
where $0 < \theta < 1$.

Relation (3) is due to Maclaurin.

Some consequences of Lagrange's M.V.T.

(1) Suppose that the function $f : [a,b] \to \mathbb{R}$ is continuous over the closed interval $[a,b]$, derivable in the open interval (a,b) and $f'(x) = 0 \ \forall x \in (a,b)$. Then f is constant throughout the interval $[a,b]$. In fact $f(x) = f(a) \ \forall x \in [a,b]$.

Proof Let $x_1, x_2 \in [a,b]$ such that $x_1 < x_2$. Then it's clear that $f(x)$ satisfies all the conditions of Lagrange's Mean value theorem in $[x_1, x_2]$ and so $\exists$ a point $\xi \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi) \quad \text{--- (1)}$$

~~From this the result follows~~ *-

Note The converse of the above theorem is not true. In fact a strictly increasing (decreasing) derivable function may have a derivative that vanishes at certain points.

If $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$, $x \in \mathbb{R}$ then f is increasing in every interval of $\mathbb{R}$. Yet $f'(0) = 0$.

Theorem If f is continuous at $x = c$ and $\lim_{x \to c} f'(x) = l$, then f is derivable at $x = c$ and $f'(c) = l$.

Proof The condition that $f'(x) \to l$ as $x \to c+$ implies the existence of a $h > 0$ such that $f'(x)$ exists at each point of $(c, c+h]$. As existence of derivative at a point $\Rightarrow$ the continuity of the function

at that point, it follows that $f(x)$ is continuous at all points of $(c, c+h]$. As the function $f(x)$ is also continuous at $x = c$, it follows that it is continuous in $[c, c+h]$. Let $y \in (c, c+h]$. Then applying Lagrange's M.V.T. to $f(x)$ in $[c, y]$ we obtain

$$f(y) - f(c) = (y - c) f'(\xi) \quad \text{where } c < \xi < y.$$

i.e $$\frac{f(y) - f(c)}{y - c} = f'\{c + \theta(y - c)\} \quad \text{where } 0 < \theta < 1$$

$$\therefore \lim_{y \to c+} \frac{f(y) - f(c)}{y - c} = \lim_{y \to c+} f'\{c + \theta(y - c)\}.$$

$$= \lim_{x \to c+} f'(x) = l.$$

So, $Rf'(c) = l$.

Similarly, $Lf'(c) = l$. Hence $f'(c) = l$.

Cauchy's Mean Value Theorem Suppose the functions $f(x)$ and $g(x)$ are (i) both continuous in the closed interval $[a, b]$, (ii) both derivable in the open interval (a, b) and (iii) $g'(x) \neq 0$ in (a, b). Then $\exists$ a point $\xi \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}.$$

Proof We construct a function $\phi : [a,b] \to \mathbb{R}$ as follows:-

$$\phi(x) = f(x) + A g(x), \quad x \in [a,b]$$

where A is a constant to be determined in such a way that (chosen).

$$\phi(a) = \phi(b)$$

i.e $f(a) + A g(a) = f(b) + A g(b)$

i.e $$-A = \frac{f(b)-f(a)}{g(b)-g(a)}$$

We note that $g(a) \neq g(b)$, because if $g(a) = g(b)$ then $g(x)$ satisfies all the conditions of Rolles theorem in $[a,b]$ and in that case $\exists$ a point $c \in (a,b)$ such that $g'(c) = 0$ which is not possible.

Now, $\phi(x)$ being the sum of two continuous functions is continuous in $[a,b]$. Also,

$$\phi'(x) = f'(x) + A g'(x)$$

exists in (a,b). Also, $\phi(a) = \phi(b)$. In other words $\phi(x)$ satisfies all the conditions of Rolles theorem in $[a,b]$ and so $\exists$ a point $\xi \in (a,b)$ such that

$$0 = \phi'(\xi) = f'(\xi) + A g'(\xi)$$

i.e $$\frac{f'(\xi)}{g'(\xi)} = -A = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

(Statement only)

Taylor's Theorem with Lagrange's form of Remainder

Statement (Suppose that a function $f(x)$ is such that

(i) the $(n-1)$th derivative f^{n-1} of f is continuous in $[a, a+h]$ and (ii) the nth derivative f^n of f exists in the open interval $(a, a+h)$.

Then $\exists$ a $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\underline{|2}} f''(a) + \dots + \frac{h^{n-1}}{\underline{|n-1}} f^{n-1}(a) + \frac{h^n}{\underline{|n}} f^n(a+\theta h)$$

Proof The hypothesis (i) of the theorem implies the existence and the continuity of

$$f, f', f'', \dots, f^{n-2}, f^{n-1}$$

in $[a, a+h]$.

We construct a function ϕ in $[a, a+h]$ as follows:-

$$\phi(x) = f(x) + (a+h-x) f'(x) + \frac{(a+h-x)^2}{\underline{|2}} f''(x) + \dots + \frac{(a+h-x)^{n-1}}{\underline{|n-1}} f^{n-1}(x) + (a+h-x)^n . A$$

where A is a constant so chosen that

$$\phi(a) = \phi(a+h)$$

(Statement only)

Taylor's Theorem with Cauchy's form of Remainder: Suppose that a function $f(x)$ is such that (i) $f^{n-1}(x)$ is continuous in $[a, a+h]$ and (ii) $f^{n}(x)$ exists in $(a, a+h)$.

Then $\exists$ a $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + \frac{h^n (1-\theta)^{n-1}}{\lfloor n-1} f^n(a+\theta h)$$

Proof: The hypothesis (i) of the theorem implies the existence and the continuity of

$$f, f', f'', \ldots, f^{n-1}$$

in $[a, a+h]$.

We construct a function $\phi(x)$ defined in $[a, a+h]$ as follows:–

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{\lfloor 2} f''(x) + \cdots + \frac{(a+h-x)^{n-2}}{\lfloor n-2} f^{n-2}(x) + \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^{n-1}(x) + (a+h-x)A$$

where A is a constant to be chosen in such a way that

$$\phi(a) = \phi(a+h)$$

Then,

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a) + hA.$$

i.e. $hA = f(a+h) - f(a) - hf'(a) - \cdots - \frac{h^{n-1}}{\lfloor n-1} f^{n-1}(a)$.

Now, as $f, f', f'', \ldots, f^{n-1}$ are all continuous in $[a, a+h]$ and $(a+h-x), (a+h-x)^2, \ldots, (a+h-x)^{n-1}$ are continuous everywhere it follows that the function $\phi(x)$ is continuous in $[a, a+h]$.

Further, because $f, f', \ldots, f^{n-1}$ are derivable in $(a, a+h)$ and the functions $(a+h-x), (a+h-x)^2, \ldots, (a+h-x)^{n-1}$ are derivable everywhere, it follows that the function $\phi(x)$ is derivable in $(a, a+h)$. In fact

$$\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)f''(x) + \cdots + \frac{(a+h-x)^{n-2}}{\lfloor n-2} f^{n-1}(x) - \frac{(a+h-x)^{n-2}}{\lfloor n-2} f^{n-1}(x) + \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^n(x) - A$$

$$= \frac{(a+h-x)^{n-1}}{\lfloor n-1} f^n(x) - A.$$

Lastly, $\phi(a) = \phi(a+h)$.

In other words, the function $\phi(x)$ satisfies all the conditions of Rolle's theorem in $[a, a+h]$ and so $\exists$ a $\theta \in (0,1)$ such that

$$\phi'(a+\theta h) = 0$$

i.e. $A = \frac{1}{\lfloor n-1} \{(a+h) - (a+\theta h)\}^{n-1} f^n(a+\theta h)$

Hence the geometrical interpretation of Rolle's theorem is "If the graph of $y = f(x)$ is represented by the arc AB without any break on $[a, b]$ having everywhere a tangent and ordinate of A and B points are same, then there exist at least one point C, on the arc AB, where the tangent is parallel to the x-axis."

Illustrative Examples.

Ex. 1. *Verify Rolle's theorem in each of the following cases :*

(i) $f(x) = \cos x$ *in* $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(ii) $f(x) = 2x^3 + x^2 - 4x - 2,\ -\sqrt{2} \le x \le \sqrt{2}$

(iii) $f(x) = 2 + (x-1)^{2/3},\ 0 \le x \le 2$

(iv) $f(x) = |x|,\ -1 \le x \le 1.$

(i) We know the function $\cos x$ is continuous and derivable for all real values of x.

$\therefore$ $f(x)$ is continuous on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and derivable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Now $f'(x) = -\sin x, -\frac{\pi}{2} < x < \frac{\pi}{2}$. Also $f\left(-\frac{\pi}{2}\right) = 0 = f\left(\frac{\pi}{2}\right)$.

Thus $f(x)$ satisfies all the conditions of Rolle's theorem.

By Rolle's theorem, we should have

$$f'(c) = 0 \text{ where } -\frac{\pi}{2} < c < \frac{\pi}{2}.$$

Here $f'(c) = 0 \Rightarrow -\sin c = 0 \Rightarrow \sin c = 0$ whose one solution is $c = 0$ and $-\frac{\pi}{2} < 0 < \frac{\pi}{2}$.

Hence, Rolle's theorem is verified for the given function.

(ii) Since every polynomial in x is continuous and derivable for every real values of x, so $f(x) = 2x^3 + x^2 - 4x - 2$ is continuous on $[-\sqrt{2}, \sqrt{2}]$ and derivable on $(-\sqrt{2}, \sqrt{2})$.

Also, $f'(x) = 6x^2 + 2x - 4, -\sqrt{2} < x < \sqrt{2}$.

Moreover,

$f(-\sqrt{2}) = 2(-\sqrt{2})^3 + (-\sqrt{2})^2 - 4(-\sqrt{2}) - 2 = 0$ and $f(\sqrt{2}) = 0$.

Thus $f(x)$ satisfies all the conditions of Rolle's theorem.

By Rolle's theorem, we should have

$$f'(c) = 0, \text{ where } -\sqrt{2} < c < \sqrt{2}.$$

Here $f'(c) = 0 \Rightarrow 6c^2 + 2c - 4 = 0$

i.e., $2(c+1)(3c-2) = 0$ whose two solutions are $c = -1, \frac{2}{3}$ and $-\sqrt{2} < -1 < \sqrt{2}$ as well as $-\sqrt{2} < \frac{2}{3} < \sqrt{2}$.

Hence, Rolle's theorem is verified for the given function.

(iii) Here $f(x) = 2 + (x-1)^{\frac{2}{3}}$ is continuous on $0 \le x \le 2$ but $f'(x) = \frac{2}{3}(x-1)^{-\frac{1}{3}}$ does not exist at $x = 1$.

Hence $f(x)$ is not derivable on $0 < x < 2$.

Thus the conditions of Rolle's theorem do not hold. So Rolle's theorem is not applicable to the given function.

(iv) The given function can be written as

$$f(x) = -x, \text{ when } -1 \le x \le 0$$
$$= x, \text{ when } 0 < x \le 1.$$

Obviously $f(x)$ is continuous on $[-1, 1]$.

Now $Lf'(0) = \lim_{h \to 0-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0-} \frac{-h-0}{h} = -1$

$Rf'(0) = \lim_{h \to 0+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0+} \frac{h-0}{h} = 1.$

$\therefore$ $Lf'(0) \ne Rf'(0)$.

So $f'(x)$ does not exist at $x = 0$ which lies between -1 and 1.

Hence $f(x)$ is not derivable on $-1 < x < 1$.

Thus the conditions of Rolle's theorem do not hold. So Rolle's theorem is not applicable to the given function.

Ex. 2. *Show that Rolle's theorem is not applicable to* $f(x)=\tan x$ *in* $[0,\pi]$, *although* $f(0)=f(\pi)$.

Here $f(x)=\tan x$ is continuous everywhere in $[0,\pi]$ except at $x=\frac{\pi}{2}$ and consequently is not derivable there.

Thus the condition of Rolle's theorem do not hold. Hence Rolle's theorem is not applicable to the function $f(x)=\tan x$ on the interval $[0,\pi]$, although $f(0)=f(\pi)$.

4.3. Lagrange's Mean Value Theorem.

Let f be a function defined on a finite closed interval $[a, b]$ such that

(i) $f(x)$ is continuous for all x, $a\le x\le b$

(ii) $f'(x)$ exists for all x, $a<x<b$.

Then there exist at least one value c, $a<c<b$, such that

$$\frac{f(b)-f(a)}{b-a}=f'(c).$$

Proof: Beyond the scope of the book.

Remark. When $f(b)=f(a)$ then $f'(c)=0, a<c<b$. So Lagrange's mean value theorem becomes Rolle's theorem when $f(b)=f(a)$.

Another form of Lagrange's Mean Value Theorem.

Let f be a function defined on a finite closed interval $[a, a+h]$ such that (i) f is continuous on $[a, a+h]$

(ii) f is derivable on $(a, a+h)$.

Then $f(a+h)=f(a)+hf'(a+\theta h),\ 0<\theta<1.$ . . . (1)

Proof. Beyond the scope of the book.

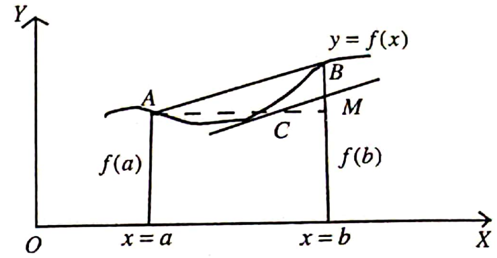
Note. For the interval $[0, h]$, (1) reduces to

$$f(h)=f(0)+hf'(\theta h),\ 0<\theta<1.$$

Geometrical Interpretation of M. V. Theorem.

From the fig. it is clear that $\frac{f(b)-f(a)}{b-a}=\frac{BM}{AM}=$ gradient of the chord AB.

On the other hand $f'(c)$ is the gradient of the tangent to the curve $y=f(x)$ at the point $(c, f(c))$.



Hence the geometrical interpretation of Lagrange's mean value theorem is "If the graph of $y=f(x)$ is represented by arc AB without any break on $[a, b]$ having tangent at every point on AB, then there exist at least one point C on arc AB, where the tangent is parallel to the chord joining the two points A and B, where $A=(a, f(a))$, $B=(b, f(b))$."

Illustrative Examples.

Ex. 1. *Verify Lagrange's mean value theorem for the following functions:*

(i) $f(x)=x(x-1)(x-2),\ 0\le x\le\frac{1}{2}$

(ii) $f(x)=x\cos\frac{1}{x}$, *for* $x\ne0$; $=0$, *for* $x=0$ in $[-1, 1]$

(iii) $f(x)=\log x,\ \frac{1}{2}\le x\le 2.$

(i) Since every polynomial in x is continuous and derivable for every real values of x, so

$$f(x)=x(x-1)(x-2)=x^3-3x^2+2x$$

is continuous on $[0,\frac{1}{2}]$ and derivable on $(0,\frac{1}{2})$.

Also $f'(x)=3x^2-6x+2,\ 0<x<\frac{1}{2}$.

Thus $f(x)$ satisfies all the conditions of Lagrange's mean value theorem.

By M. V. Theorem we should have an c, $0<c<\frac{1}{2}$, such that

$$f'(c) = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}.$$

Here $f'(c) = \dfrac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$ implies

$$3c^2 - 6c + 2 = 2\left[\left(\frac{1}{2}\right)^3 - 3.\left(\frac{1}{2}\right)^2 + 2.\frac{1}{2} - 0\right]$$

or, $3c^2 - 6c + 2 = \frac{3}{4}$ or, $12c^2 - 24c + 5 = 0$

having two solution $c = 1 \pm \frac{\sqrt{21}}{6}$. Between these two $1 - \frac{\sqrt{21}}{6}$ lies between 0 and $\frac{1}{2}$.

Hence, Lagrange's mean value theorem is verified for the given function.

(ii) Since x and $\cos\left(\frac{1}{x}\right)$ are derivable for all $x \neq 0$, so $f(x)$ is derivable for all $x \neq 0$.

$$\therefore \quad f'(x) = \frac{d}{dx}\left\{x\cos\left(\frac{1}{x}\right)\right\}$$

$$= \cos\left(\frac{1}{x}\right) - \frac{1}{x}\sin\frac{1}{x} \text{ when } x \neq 0.$$

But $f'(0) = \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$

$$= \lim_{h \to 0} \frac{h\cos\frac{1}{h} - 0}{h} = \lim_{h \to 0} \cos\frac{1}{h} \text{ which does not exist.}$$

Hence $f(x)$ is not derivable on $(-1, 1)$.

Thus Lagrange's mean value theorem is not applicable for the given function.

(iii) Since $\log x$ is continuous and derivable for all x in $(0, \infty)$, so $f(x) = \log x$ is continuous on $\frac{1}{2} \leq x \leq 2$ and derivable on $\frac{1}{2} < x < 2$.

Also $f'(x) = \frac{1}{x}$, $\frac{1}{2} < x < 2$.

Thus $f(x)$ satisfies all the conditions of Lagrange's mean value theorem.

$\therefore$ There should exist c, $\frac{1}{2} < c < 2$, such that

$$f'(c) = \frac{f(2) - f\left(\frac{1}{2}\right)}{2 - \frac{1}{2}}.$$

Here $f'(c) = \dfrac{f(2) - f\left(\frac{1}{2}\right)}{2 - \frac{1}{2}} \Rightarrow \dfrac{1}{c} = \dfrac{\log 2 - \log\frac{1}{2}}{\frac{3}{2}} = \dfrac{2}{3}\log 4$

which gives $c = \dfrac{3}{2\log 4}$ which lies between $\frac{1}{2}$ and 2.

Ex. 2. *If $f'(x) = 0$ in $[a, b]$, then by using mean value theorem, show that $f(x)$ is constant in that interval.*

Let x_1, x_2 be two arbitrary points in $[a, b]$ such that $a \leq x_1 < x_2 \leq b$. Then applying Lagrange's mean value theorem to $f(x)$ in $[x_1, x_2]$

we get, $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad x_1 < c < x_2,$

$= 0$ by the given condition.

$\therefore \quad f(x_2) - f(x_1) = 0 \qquad \therefore \quad f(x_2) = f(x_1).$

Since x_1, x_2 are any two arbitrary points in $[a, b]$, so it follows that $f(x)$ is constant in $[a, b]$.

Ex. 3. *If $f'(x)$ exists and < 0 everywhere in (a, b), then by using mean value theorem show that $f(x)$ is a decreasing function in (a, b).*

Let x_1, x_2 be two arbitrary points in (a, b) such that $a < x_1 < x_2 < b$. Then applying Lagrange's mean value theorem to $f(x)$ in $[x_1, x_2]$, we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \; x_1 < c < x_2.$$

$\therefore \quad \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} < 0$, [$\because f'(c) < 0$ by the given hypothesis.]

But $x_2 - x_1 > 0$. So from above $f(x_2) - f(x_1) < 0$

i.e., $f(x_2) < f(x_1)$ whenever $x_2 > x_1$.

Hence $f(x)$ is a decreasing function in (a, b).

Ex. 4. *If $f(x)=a+bx+c\,m^x$, show that θ is independent of x in the mean value theorem $f(x+h)=f(x)+h\,f'(x+\theta h)$*

Given $f(x)=a+bx+c\,m^x$ $\quad\therefore\; f'(x)=b+c\,m^x\log m.$

$\therefore$ From $f(x+h)=f(x)+h\,f'(x+\theta h)$, we have

$$a+b(x+h)+c\,m^{x+h}=a+bx+cm^x+h\{b+cm^{x+\theta h}\log m\}$$

or, $c\,m^x.m^h=c\,m^x+c.m^x m^{\theta h}\log m$

or, $m^h=1+m^{\theta h}\log m$ $\quad$ or, $m^{\theta h}=\dfrac{m^h-1}{\log m}$

or, $\theta\,h\log m=\log\left(\dfrac{m^h-1}{\log m}\right)$

$\therefore$ $\theta=\dfrac{1}{h\log m}\log\left(\dfrac{m^h-1}{\log m}\right)$, which is independent of x.

Ex. 5. *In the mean value theorem $f(h)=f(0)+hf'(\theta h),\,0<\theta<1$ show that the limiting value of θ as $h\to0+$ is $\frac{1}{2}$ if $f(x)=\cos x$.*

Here $f(x)=\cos x$ $\therefore$ $f(0)=1$ *and* $f'(x)=-\sin x.$

$\therefore$ from $f(h)=f(0)+h\,f'(\theta h)$, we have

$\cos h=1+h(-\sin\theta h)$ $\quad$ or, $\sin\theta h=\dfrac{1-\cos h}{h}$

or, $\theta.\dfrac{\sin\theta h}{\theta h}=\dfrac{1}{2}\left(\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$

$\therefore$ $\lim\limits_{h\to0+}\left(\theta.\dfrac{\sin\theta h}{\theta h}\right)=\dfrac{1}{2}\lim\limits_{h\to0+}\left(\dfrac{\sin\frac{h}{2}}{\frac{h}{2}}\right)^2$

$\therefore$ $\lim\limits_{h\to0+}\theta\times1=\frac{1}{2}\cdot(1)^2.$ $\therefore$ $\lim\limits_{h\to0+}\theta=\frac{1}{2}.$

Ex. 6. *Use Mean-value theorem to prove the following inequalities :*

(i) $0<\dfrac{1}{x}\log\dfrac{e^x-1}{x}<1.$

(ii) $\dfrac{x}{1+x}<\log(1+x)<x$ *if* $x>0.$

(i) Let $f(x)=e^x$.

Then from the mean value theorem

$$f(x)=f(0)+x\,f'(\theta x),\;0<\theta<1,$$

we have $e^x=e^0+x\,e^{\theta x}$ $\;[\because f'(x)=e^x]$

or, $e^x-1=x\,e^{\theta x}$, $\quad$ or, $e^{\theta x}=\dfrac{e^x-1}{x}$

or, $\theta x=\log\dfrac{e^x-1}{x}$ $\quad$ or, $\theta=\dfrac{1}{x}\log\dfrac{e^x-1}{x}$,

$\therefore$ $0<\dfrac{1}{x}\log\dfrac{e^x-1}{x}<1.$ $\;[\because 0<\theta<1].$

(ii) Let $f(x)=\log(1+x)$.

Then from the mean value theorem,

$$f(x)=f(0)+x\,f'(\theta x),\;0<\theta<1,$$

we have $\log(1+x)=\log1+x.\dfrac{1}{1+\theta x}$ $\left[\because f'(x)=\dfrac{1}{1+x}\right]$

$\therefore$ $\log(1+x)=\dfrac{x}{1+\theta x}$ $\quad\ldots$ (1)

Since $0<\theta<1$ and $x>0$, so $0<\theta x<x.$

or, $1<1+\theta x<1+x.$ $\quad$ or, $1>\dfrac{1}{1+\theta x}>\dfrac{1}{1+x}$

or, $x>\dfrac{x}{1+\theta x}>\dfrac{x}{1+x}$ $\quad$ or, $\dfrac{x}{1+x}<\dfrac{x}{1+\theta x}<x.$

$\therefore$ $\dfrac{x}{1+x}<\log(1+x)<x,$ by (1).

4.4. Cauchy's Mean Value Theorem.

Let f and g be two functions defined on $[a,b]$ such that

(i) f and g are continuous on $[a,b]$

(ii) f and g are derivable on (a,b), and

(iii) $g'(x)\neq0$ for any x in (a,b).

Then there exist at least one value c, $a<c<b$, such that

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}. \qquad \cdots \quad (1)$$

Proof: Beyond the scope of the book.

Lagrange's M. V. Theorem From Cauchy's M. V. Theorem.

The function x satisfies the conditions satisfied by the 2nd function $g(x)$ in Cauchy's M. V. Th. If we take $g(x)=x$ we have

$$\frac{f(b)-f(a)}{b-a}=\frac{f'(c)}{1} \quad [\because g'(x)=1]$$

i.e., $$\frac{f(b)-f(a)}{b-a}=f'(c),$$

which is the result of Lagrange's Mean value theorem.

Another Form of Cauchy's M. V. Theorem.

Let $b=a+h$. Then $c=a+\theta h$, $0<\theta<1$.

So (1) reduces to

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)}=\frac{f'(a+\theta h)}{g'(a+\theta h)}, \quad 0<\theta<1.$$

which is the another form of Cauchy's Mean Value theorem.

Illustrative Examples.

Ex. 1. *Verify Cauchy's Mean value theorem, for the following pairs of functions*

(i) $f(x)=\sin x, g(x)=\cos x, \; x\in\left[-\frac{\pi}{2},0\right]$

(ii) $f(x)=\sqrt{x}, \; g(x)=\frac{1}{\sqrt{x}}, \; x\in[1,2]$.

(i) Since $\sin x$ and $\cos x$ are both continuous and derivable for all $x\in R$ so $f(x)=\sin x$, $g(x)=\cos x$ are continuous on $\left[-\frac{\pi}{2},0\right]$ and derivable on $\left(-\frac{\pi}{2},0\right)$.

Also $f'(x)=\cos x$, $g'(x)=-\sin x$. Further $g'(x)\neq 0$ for any x in $\left(-\frac{\pi}{2},0\right)$.

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem.

According to Cauchy's M. V. Theorem there should exist c, $-\frac{\pi}{2}<c<0$, such that

$$\frac{f(0)-f\left(-\frac{\pi}{2}\right)}{g(0)-g\left(-\frac{\pi}{2}\right)}=\frac{f'(c)}{g'(c)}.$$

Here $$\frac{f(0)-f\left(-\frac{\pi}{2}\right)}{g(0)-g\left(-\frac{\pi}{2}\right)}=\frac{f'(c)}{g'(c)} \Rightarrow \frac{0-(-1)}{1-0}=\frac{\cos c}{-\sin c}$$

or, $\cot c=-1$ which gives a solution $c=-\frac{\pi}{4}$ which lies between $-\frac{\pi}{2}$ and 0.

Hence, Cauchy's Mean Value theorem is verified.

(ii) Since, the power function x^n, where n is a rational number, is continuous and derivable for all $x>0$, so $f(x)=\sqrt{x}$, $g(x)=\frac{1}{\sqrt{x}}$ are continuous on [1, 2] and derivable on (1, 2).

Also $f'(x)=\frac{1}{2\sqrt{x}}$, $g'(x)=-\frac{1}{2x^{3/2}}$.

Further we see $g'(x)\neq 0$ for any x in (1, 2).

Thus f and g satisfy all the conditions of Cauchy's Mean Value Theorem.

There should exist c, $1<c<2$, such that

$\frac{f(2)-f(1)}{g(2)-g(1)}=\frac{f'(c)}{g'(c)}$. Here this relation implies

$$\frac{\sqrt{2}-1}{\frac{1}{\sqrt{2}}-1}=\frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2c^{3/2}}}$$ whose solution is $c=\sqrt{2}$ which lies between 1 and 2.

Hence Cauchy's Mean Value theorem is verified.

Ex. 2. *In Cauchy's Mean Value theorem, if $f(x) = e^x$ and $g(x) = e^{-x}$, show that θ is independent of both x and h and is equal to $\frac{1}{2}$.*

Cauchy's Mean value theorem is

$$\frac{f(x+h)-f(x)}{g(x+h)-g(x)} = \frac{f'(x+\theta h)}{g'(x+\theta h)}, \; 0<\theta<1.$$

$$\therefore \quad \frac{e^{x+h}-e^x}{e^{-(x+h)}-e^{-x}} = \frac{e^{x+\theta h}}{-e^{-(x+\theta h)}} \; [\because f'(x) = e^x, g'(x) = -e^{-x}]$$

$$\text{or,} \quad \frac{e^x(e^h-1)}{e^{-x}(e^{-h}-1)} = \frac{e^x . e^{\theta h}}{-e^{-x} . e^{-\theta h}}$$

$$\text{or,} \quad \frac{e^h(e^h-1)}{1-e^h} = -\frac{e^{\theta h}}{e^{-\theta h}} \quad \text{or,} -e^h = -e^{2\theta h}$$

$$\text{or,} \quad 2\theta h = h \; \therefore \theta = \frac{1}{2}.$$

So θ is independent of both x and h and is equal to $\frac{1}{2}$.

4.5. Generalised Mean-Value theorem : Taylor's Theorem.

Theorem-1. (***Taylor's theorem with Lagrange's form of remainder***)

Let f be a function defined on $[a, b]$ such that

(i) the $(n-1)$th derivative $f^{(n-1)}$ is continuous on $[a, b]$, and

(ii) the nth derivative $f^{(n)}$ exists on (a, b).

Then there exist at least one value c, $a<c<b$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f''(a) + \cdots + \frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a) + \frac{1}{n!}(b-a)^n f^{(n)}(c). \quad \ldots (1)$$

Proof. Beyond the scope of the book.

Another form of the above Theorem.

Let f be a function defined on $[a, a+h]$, $h>0$ such that

(i) $f^{(n-1)}$ is continuous on $[a, a+h]$, and

(ii) $f^{(n)}$ exists on $(a, a+h)$.

Then there exist at least one number $\theta, 0<\theta<1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h). \quad \ldots (2)$$

Proof. Beyond scope of this book.

Note. The last term of the above series i.e., $(n+1)$ th term, is called the Remainder after n terms and is denoted by R_n.

$$\therefore \quad R_n = \frac{h^n}{n!}f^n(a+\theta h), 0<\theta<1.$$

Theorem 2. ***Taylor's theorem with Cauchy's form of remainder.***

Let f be a function defined on $[a, b]$ such that

(i) $f^{(n-1)}$ is continuous on $[a, b]$, and

(ii) $f^{(n)}$ exists on (a, b).

Then there exist at least one value c, $a<c<b$, such that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2!}(b-a)^2 f''(a) + \cdots + \frac{1}{(n-1)!}(b-a)^{n-1} f^{(n-1)}(a) + \frac{1}{(n-1)!}(b-a)(b-c)^{n-1} f^{(n)}(c). \quad (3)$$

Proof. Beyond scope of the book.

Another form of the above theorem.

Let f be a function defined on $[a, a+h]$, $h>0$ such that

(i) $f^{(n-1)}$ is continuous on $[a, a+h]$, and

(ii) $f^{(n)}$ exists on $(a, a+h)$.

Then there exist at least one number θ, $0<\theta<1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(a+\theta h). \quad \ldots (4)$$

Proof. Beyond the scope of the book.

Note. The last term of the above series i.e, the $(n+1)$th term, is called the Remainder after n terms and is denoted by R_n.

$$\therefore \quad R_n = \frac{h^n(1-\theta)^{n-1}}{(n-1)!} f^n(a+\theta h), 0<\theta<1.$$

Remarks. (i) The conclusion of Taylor's theorem, is also known as *Taylor's formula* or *Taylor's series* of the function $f(x)$.

(ii) By putting $b = x$ in (1), we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots$$

$$+ \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}\{a+\theta(x-a)\}, \quad 0<\theta<1$$

which is called the *expansion* of $f(x)$ about $x = a$.

(iii) Taylor's theorem is also called the *Mean Value theorem of the nth order.* So the 3rd order mean value theorem is

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a+\theta h), 0<\theta<1.$$

4.6. Maclaurin's Theorem.

Let f be a function defined on $[-h, h]$, $h > 0$ such that

(i) $f^{(n-1)}$ is continuous on $[-h, h]$, and

(ii) $f^{(n)}$ exists on $(-h, h)$.

Then there exist at least one number $\theta, 0<\theta<1$, such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

$$\cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x).$$

Proof. Beyond the scope of the book.

Note. The above series is known as *Maclaurin's series in finite form* for the function $f(x)$.

Illustrative Examples.

Ex. 1. *Write a Taylor's formula for the function*

$f(x) = \log(1+x), -1<x<\infty$ about $x = 2$ with Lagranges form of remainder after 3 terms.

Here the function $f(x) = \log(1+x), -1<x<\infty$ is continuous and derivable at each point of $(-1, \infty)$.

Now, $f'(x) = \dfrac{1}{1+x}$, $f''(x) = -\dfrac{1}{(1+x)^2}$, $f'''(x) = \dfrac{2}{(1+x)^3} \ldots$ and so on.

Now the Taylor's formula for the function $f(x)$ about $x = 2$ is

$$f(x) = f(2) + (x-2) f'(2) + \frac{(x-2)^2}{2!} f''(2)$$

$$+ \frac{(x-2)^3}{3!} f'''\{2+\theta(x-2)\}, 0<\theta<1.$$

Since $f(2) = \log 3$, $f'(2) = \dfrac{1}{3}$, $f''(2) = -\dfrac{1}{9}$.

$$\therefore \quad f(x) = \log 3 + \frac{1}{3}(x-2) + \frac{1}{2!}(x-2)^2\left(-\frac{1}{9}\right)$$

$$+ \frac{(x-2)^3}{3!} \frac{2}{\{1+2+\theta(x-2)\}^3}$$

$$\therefore \log(1+x) = \log x + \frac{1}{3}(x-2) - \frac{1}{18}(x-2)^2 + \frac{(x-2)^3}{3} \frac{1}{\{3+\theta(x-2)\}^3},$$

where $0<\theta<1$.

Ex. 2. *Expand* $\cos x$ *in a finite series (in power of x) in Lagrange's form of remainder.*

Let $f(x) = \cos x \quad \therefore f(0) = 1$.

Now, $f^{(n)}(x) = \cos\left(n\dfrac{\pi}{2} + x\right)$. $\quad \therefore \quad f^{(n)}(0) = \cos\dfrac{n\pi}{2}$.

$\therefore \quad f'(0) = 0, f''(0) = -1, f'''(0) = 0, f^{iv}(0) = 1$ and so on.

Now the Maclaurin's series for the function $f(x)$ in finite form in Lagrange's form of remainder, is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{iv}(0)$$

$$+ \cdots + \frac{x^n}{n!} f^n(\theta x), \; 0<\theta<1$$

$\therefore \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!}\cos\left(\frac{n\pi}{2} + \theta x\right).$

Ex. 3. *Show that* $(x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}h + \frac{3.1}{2.2}\frac{h^2}{2!}\frac{1}{\sqrt{x+\theta h}}, 0 < \theta < 1.$

Find θ *when* $x = 0$.

Let $f(x) = x^{\frac{3}{2}}$.

$\therefore \quad f'(x) = \frac{3}{2}x^{\frac{1}{2}}, f''(x) = \frac{3}{2}\cdot\frac{1}{2}\cdot x^{-\frac{1}{2}}.$

Now the 2nd order Mean Value theorem states

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x+\theta h), 0 < \theta < 1.$$

$$\therefore \quad (x+h)^{\frac{3}{2}} = x^{\frac{3}{2}} + h.\frac{3}{2}x^{\frac{1}{2}} + \frac{h^2}{2!}\frac{3}{2}\cdot\frac{1}{2}(x+\theta h)^{-\frac{1}{2}}$$

$$= x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{1}{2}}h + \frac{3.1}{2.2}\frac{h^2}{2!}\cdot\frac{1}{\sqrt{x+\theta h}}.$$

Putting $x = 0$, we get $h^{\frac{3}{2}} = \frac{3}{8}\frac{h^2}{\sqrt{\theta h}}$. $\quad \therefore \ \theta = \frac{9}{64}$.

Ex. 4. *Using Mean Value theorem, show that*

$$\sin x > x - \frac{1}{6}x^3, \ \ if \ 0 < x < \frac{\pi}{2}.$$

Let $f(x) = \sin x$ $\qquad \therefore f(0) = 0$

$f'(x) = \cos x$ $\qquad \therefore f'(0) = 1$

$f''(x) = -\sin x$ $\qquad \therefore f''(0) = 0$

$f'''(x) = -\cos x$ $\qquad \therefore f'''(\theta x) = -\cos\theta x.$

Now the Maclaurin's series for the function $f(x)$ of order 3 is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x), \ 0 < \theta < 1.$$

$$\therefore \quad \sin x = 0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}(-\cos\theta x)$$

or, $\sin x = x - \frac{x^3}{6}.\cos\theta x \qquad \ldots \quad (1)$

Since $0 < x < \frac{\pi}{2}$ and $0 < \theta < 1$, so $0 < \theta x < \frac{\pi}{2}$

or, $0 < \cos\theta x < 1$.

$\therefore \quad -\frac{x^3}{6}\cos\theta x > -\frac{x^3}{6}, \qquad \because x > 0$

or, $x - \frac{x^3}{6}\cos\theta x > x - \frac{x^3}{6}$.

$\therefore \quad \sin x > x - \frac{x^3}{6}$, by (1).

Ex. 5. *Show that if* $-1 < x < 1$,

$$(1+x)^{\frac{1}{3}} = 1 + \frac{1}{3}x - \frac{1}{9}x^2 + \frac{5}{81}x^3 - \frac{10}{243}x^4 + R_5$$

where $\quad R_5 = \frac{x^5}{4!}(1-\theta)^4\left(-\frac{880}{243}\right)(1+\theta x)^{-\frac{14}{3}}.$

Let $\quad f(x) = (1+x)^{\frac{1}{3}}, -1 < x < 1. \qquad \therefore f(0) = 1.$

$\therefore \quad f'(x) = \frac{1}{3}(1+x)^{-\frac{2}{3}} \qquad \therefore f'(0) = \frac{1}{3}$

$f''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)(1+x)^{-\frac{5}{3}} \qquad \therefore f''(0) = -\frac{2}{9}$

$f'''(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)(1+x)^{-\frac{8}{3}} \qquad \therefore f'''(0) = \frac{10}{27}$

$f^{(4)}(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)(1+x)^{-\frac{11}{3}} \qquad \therefore f^4(0) = -\frac{80}{81}$

$f^{(5)}(x) = \frac{1}{3}\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)\left(-\frac{8}{3}\right)\left(-\frac{11}{3}\right)(1+x)^{-\frac{14}{3}}$

$= \frac{880}{243}(1+x)^{-\frac{14}{3}}.$

Now Taylor's Expansion of $f(x)$ about $x = 0$, with Cauchy's form of remainder, after 5 terms, is

$$f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(0)+\frac{x^3}{3!}f'''(0)$$

$$+\frac{x^4}{4!}f^{(4)}(0)+\frac{x^5}{4!}(1-\theta)^4 f^{(5)}(\theta x), 0<\theta<1.$$

$$\therefore\ (1+x)^{\frac{1}{3}}=1+\frac{1}{3}x-\frac{2}{9}\frac{x^2}{2!}+\frac{10}{27}\frac{x^3}{3!}-\frac{80}{81}\frac{x^4}{4!}+\frac{880}{243}\frac{x^5}{4!}(1-\theta)^4(1+\theta x)^{-\frac{14}{3}}$$

$$\therefore\ (1+x)^{\frac{1}{3}}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\frac{10}{243}x^4+R_5$$

$$\text{where } R_5=\frac{x^5}{4!}(1-\theta)^4\frac{880}{243}(1+\theta x)^{-\frac{14}{3}}$$

Ex. 6. *Prove that* $\sin 46^\circ \sim \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right)$. *Is this estimate high or less.*

We show two different processes.

1st Procedure (Using M. V. T.) : Let $f(x)=\sin x$. By Lagrange's M.V.T. (2nd form) we have

$$f(a+h)=f(a)+hf'(a+\theta h) \text{ where } 0<\theta<1$$

$f(a+h)-f(a)=h\,f'(a+\theta h)$. Putting $a=45^\circ, h=1^\circ$ we get from above

$$f(46^\circ)=f(45^\circ)+1^\circ\times\cos(45^\circ+\theta\times 1^\circ)$$

$$\text{or, } \sin 46^\circ=\sin 45^\circ+\frac{\pi}{180}\cos(45^\circ+\theta^\circ) \quad \ldots \quad (1)$$

$$\cong \sin 45^\circ+\frac{\pi}{180}\cos 45^\circ. \quad [\because \theta^\circ \text{ is so small}]$$

$$\therefore \quad \sin 46^\circ \cong \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right) \quad \therefore \sin 46^\circ \cong \frac{1}{2}\sqrt{2}\left(1+\frac{\pi}{180}\right).$$

For small θ°, $\cos(45^\circ+\theta^\circ)<\cos 45^\circ$. Therefore from (1), exact value of $\sin 46^\circ <$ Approx value of $\sin 46^\circ$.

This shows that the estimate is high.

2nd procedure (using Taylor's Theorem) : Let $f(x)=\sin x$. Applying 2nd order Taylor's Theorem we get

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a+\theta h).$$

We put $a=45^\circ, h=1^\circ$. Then from above we get

$$f(a+h)\cong f(a)+hf'(a)$$

$$\text{or, } f(46^\circ)\cong f(45^\circ)+1^\circ\cos 45^\circ$$

$$\text{or, } \sin 46^\circ \cong \frac{1}{\sqrt{2}}\left(1+\frac{\pi}{180}\right).$$

4.7. Maclaurin's series.

Theorem : Suppose $f(x)$ has derivatives of every order in $[-h, h]$, for some $h>0$. Then for $x\in[-h, h]$

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^n(0)+\cdots\infty$$

if $\lim R_n=0$ where R_n is the remainder after n terms.

Proof. Let S_n denote the sum of the first n terms of this series. Then by Maclaurin's theorem, we have for every point x in $[-h, h]$,

$$f(x)=S_n+R_n$$

where R_n is the remainder after n terms in the Maclaurin's theorem.

If we have $\lim_{n\to\infty} R_n=0$, then it follows at once that the Maclaurin's series has a sum equal to $f(x)$ i.e., if $\lim_{n\to\infty} R_n=0$, then

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\cdots+\frac{x^n}{n!}f^{(n)}(0)+\cdots\infty. \quad (1)$$

Note : In this case we say that $f(x)$ is represented by its Maclaurin's series (1).

An Important result.

We shall prove that $\lim_{n\to\infty}\frac{x^n}{n!}=0$ for every x. Chose a positive integer $k>|x|$. Then for all $n>2k$, we have

$$0 \le \left|\frac{x^n}{n!}\right| = \frac{|x|^n}{n!} < \frac{k^n}{n!} = \left(\frac{k}{1}\right)\left(\frac{k}{2}\right)\cdots\left(\frac{k}{2k-1}\right)\cdot\left(\frac{k}{2k}\right)\left(\frac{k}{2k+1}\right)\cdots\left(\frac{k}{n}\right)$$

$$< \frac{k^{2k-1}}{(2k-1)!}\cdot\frac{1}{2^{n-(2k-1)}} = \frac{(2k)^{2k-1}}{(2k-1)!}\cdot\frac{1}{2^n} \to 0 \text{ as } n \to \infty.$$

Hence $\lim_{n\to\infty} \frac{x^n}{n!} = 0.$

Illustrative Examples.

Ex. (Can be used as formula) ***Expand the following functions in power of x in infinite series :***

(1) $\sin x$; (2) e^x;

(3) $\log(1+x)$, $-1 < x \le 1$.

(1) Let $f(x) = \sin x$.

So its nth derivative $f^{(n)}(x) = \sin\left(\frac{n\pi}{2} + x\right)$.

Thus $f(x)$ possesses derivatives of every order for every value of x.

Also $f^{(n)}(0) = \sin\frac{n\pi}{2}$

$\therefore$ $f'(0)=1, f''(0)=0, f'''(0)=-1, f^{(4)}(0)=0, f^{(5)}(0)=1$ and so on.

Now the remainder after n terms in Maclaurin's theorem in Lagrange's form is $R_n = \frac{x^n}{n!} f^{(n)}(\theta x) = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)$, $0 < \theta < 1$.

$$\therefore \quad 0 \le |R_n| = \left|\frac{x^n}{n!}\sin\left(\frac{n\pi}{2} + \theta x\right)\right| = \left|\frac{x^n}{n!}\right| \cdot \left|\sin\left(\frac{n\pi}{2} + \theta x\right)\right|$$

$$\le \left|\frac{x^n}{n!}\right| .1 \quad [\because |\sin x| \le 1 \text{ for all } x]$$

or, $-\left|\frac{x^n}{n!}\right| \le R_n \le \left|\frac{x^n}{n!}\right|$.

Since $\lim_{n\to\infty} \frac{x^n}{n!} = 0$ for every x, so $\lim_{n\to\infty}\left|\frac{x^n}{n!}\right| = 0$. So it follows from above that $\lim_{n\to\infty} R_n = 0$, for every x.

Thus for every x, we have (the Macl. series)

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\infty.$$

$$\therefore \quad \sin x = 0 + \frac{x}{1!}.1 + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1)\cdots\infty.$$

$$\therefore \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\infty.$$

(2) Let $f(x) = e^x$. So its nth derivative $f^{(n)}(x) = e^x$.

Then $f(x)$ possesses derivative of every order for every value of x.

Also $f^{(n)}(0) = e^0 = 1$.

Now the remainder after n terms in Maclaurin's theorem in Lagrange's form is

$$R_n = \frac{x^n}{n!}f^{(n)}(\theta x) = \frac{x^n}{n!}e^{\theta x}, \quad 0 < \theta < 1.$$

$$\therefore \quad 0 \le |R_n| = \left|\frac{x^n}{n!}\right| \left|e^{\theta x}\right| \le \left|\frac{x^n}{n!}\right| . e^{|\theta x|} \le \left|\frac{x^n}{n!}\right| . e^{|x|}, \ [\because 0 < \theta < 1, |\theta x| \le |x|.]$$

$$\to 0 \text{ as } n \to \infty. \quad \left[\because \lim_{n\to\infty}\frac{x^n}{n!} = 0 \text{ for every } x\right]$$

$\therefore$ It follows that $\lim_{n\to\infty} R_n = 0$, for every x.

Thus for every x, we have,

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots\infty$$

$$= 1 + \frac{x}{1!}.1 + \frac{x^2}{2!}.1 + \frac{x^3}{3!} + \cdots\infty$$

$$\therefore \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\infty.$$

(3) Let $f(x) = \log(1+x)$. So its nth derivative

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n}.$$

Thus $f(x)$ possesses derivatives of every order for every value of x

$\therefore \quad f^{(n)}(0)=(-1)^{n-1}(n-1)!$

Case I. Let $0 \le x \le 1$.

Now the remainder after n terms in Maclaurin's theorem in Lagrange's form is

$$R_n=\frac{x^n}{n!}f^{(n)}(\theta x),\ 0<\theta<1$$

$$=\frac{x^n}{n!}\frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}=(-1)^{n-1}\frac{x^n}{n}\cdot\frac{1}{(1+\theta x)^n}.$$

$$\therefore \quad 0\le|R_n|=\frac{1}{n}\left|\frac{x^n}{(1+\theta x)^n}\right|\le\frac{1}{n}\cdot\frac{1}{(1+\theta x)^n} \quad [\because 0<x\le 1]$$

$$\le\frac{1}{n}\left[\because 0<\theta<1 \text{ and } x\ge 0, \text{ so } 0<\frac{1}{1+\theta x}\le 1\right]$$

$\to 0$ as $n\to\infty$.

$\therefore \quad \lim_{n\to\infty} R_n=0$ when $0\le x\le 1$.

Case II. Let $-1<x<0$.

Since $0<\theta<1$, so $0.x>\theta.x>1.x$ $[\because x<0]$

i.e., $1>1+\theta x>1+x$. $\therefore\ 1<\frac{1}{1+\theta x}<\frac{1}{1+x}$. ... (A)

Again $-1<x<0$. $\therefore -\theta<x\theta<0.\theta$ $(\because 0<\theta<1)$

... (B)

Now the remainder after n terms in Maclaurin's theorem in Cauchy's form is $R_n=\frac{x^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}(\theta x), 0<\theta<1$

$$=\frac{x^n(1-\theta)^{n-1}}{(n-1)!}\cdot\frac{(-1)^{n-1}(n-1)!}{(1+\theta x)^n}$$

$$=(-1)^{n-1}x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\cdot\frac{1}{1+\theta x}$$

$$\therefore \quad 0\le|R_n|=|x|^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\cdot\frac{1}{1+\theta x}\le|x|^n.1.\frac{1}{1+x} \text{ by (A) \& (B)}$$

$\to 0$ as $n\to\infty$, since $|x|<1$.

$\therefore \quad \lim_{n\to\infty} R_n=0$ when $-1<x<0$.

By case I and II, for $-1<x\le 1$, we have (the Maclaurin's series)

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(0)+\cdots\infty.$$

$$\therefore \quad \log(1+x)=\log 1+\frac{x}{1!}.1+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2!)+\cdots \text{ up to } \infty$$

$$\therefore \quad \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots \text{ up to } \infty$$

EXERCISE

[I] SHORT ANSWER QUESTIONS

1. Verify Rolle's theorem in each of the following cases :

(i) $f(x)=x,\ 0\le x<1$
$=0,\ x=1.$

(ii) $f(x)=4x-x^2,\ 0\le x\le 4.$

(iii) $f(x)=(x-1)^{2/3}+2,\ 0\le x\le 2.$

(iv) $f(x)=x^2-5x+6$ in $2\le x\le 3.$

2. If $f(x)=x^\alpha \log x,\ x>0$
$=0\ ,\ x=0$

find the value of α so that the function obeys the Rolle's theorem in [0, 1].

3. Examine the validity of (a) the hypothesis and (b) the conclusion of Lagrange's Mean value theorem for the following functions :

(i) $f(x)=x(x-1),\ 1\le x\le 2.$

(ii) $f(x)=x\sin\frac{1}{x}$, for $x\ne 0$; $=0$, for $x=0$ in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$