~~Then by first comparison test, $\sum \frac{1}{n^n}$ is convergent~~,

SEM-2, CC-3, UNIT-2

AB

Monotone sequences and their convergence

Statement : (i) A monotone increasing sequence, if bounded above, is convergent and it converges to the least upper bound i.e. supremum.

(ii) A monotone decreasing sequence, if bounded below, is convergent and it converges to the greatest lower bound i.e. infimum.

Proof (i) already done in class

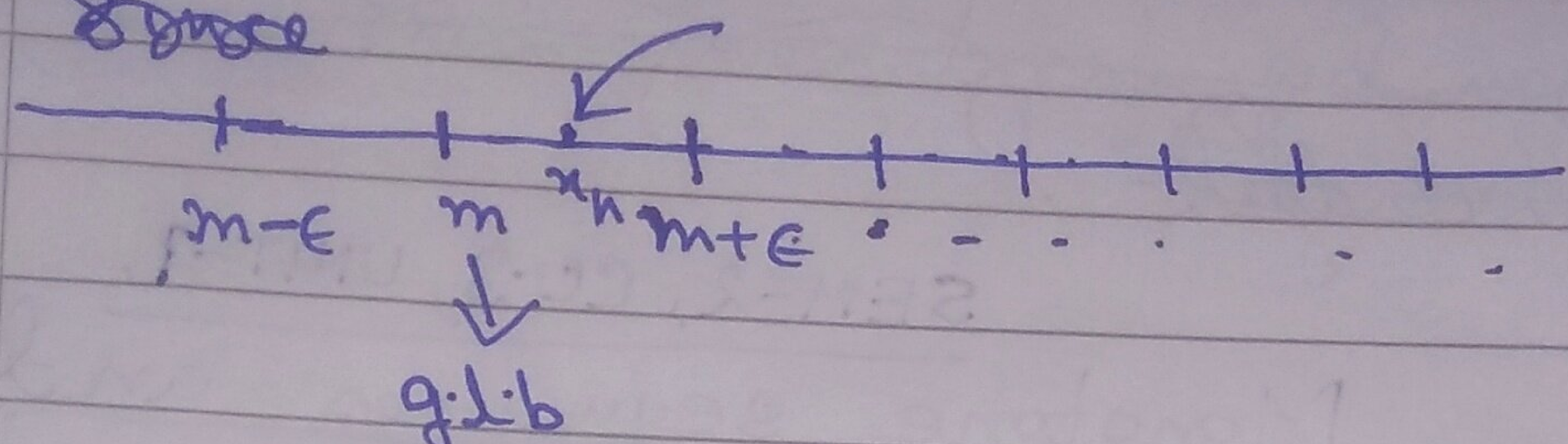
Proof (ii) :—

Let $\{x_n\}$ be a monotonic decreasing sequence and let m be its greatest lower bound.

Then i) $x_n \geq m$, $\forall n \in \mathbb{N}$

(ii) for any pre-assigned +ve ϵ, $\exists$ $k \in \mathbb{N}$, $x_k < m + \epsilon$

~~Since~~



Since $\{x_n\}$ is m.d sequence.

i.e. $m-\epsilon < m \leq x_k < x_{k+1} < \cdots\cdots < m+\epsilon$

i.e. $m-\epsilon < x_n < m+\epsilon$, $\forall n \geq k$.

$\therefore$ $\{x_n\}$ is convergent and converges to its greatest lower bound, i.e. m.

——x——

Theorem:

i) A m.i sequence that is unbounded above, diverge to ∞.

(see the proof, from Mapa)

(ii) A m.d sequence, that is unbounded below, diverge to $-\infty$.

Proof: Let $\{x_n\}$ be a m.d sequence, unbounded below,

Then for any arb., +ve no. G, however large, $\exists$ a natural no k such that

$$x_k < -G.$$

Since the sequence $\{x_n\}$ is m.d,

$$\therefore \cdots\cdots < x_{k+2} < x_{k+1} < x_k < -G$$

$$\therefore \; x_n < -G, \; \forall n \geq k$$

Hence $\{x_n\}$ diverges to $-\infty$.

very Important

Cantor's Intersection Theorem (Nested Interval Theorem)

Statement :-

Let $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals such that

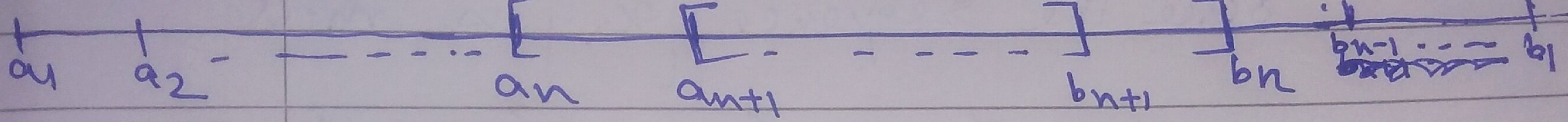
(i) $[a_{n+1}, a_{n+1}] \subset [a_n, b_n]$, $\forall n \in \mathbb{N}$ and

(ii) $\lim (b_n - a_n) = 0$, ~~$\forall n \in \mathbb{N}$~~

Then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains exactly one and one element.

Proof ; Now, $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$, $\forall n \in \mathbb{N}$

$\therefore$ $a_n \leq a_{n+1}$, $b_{n+1} \leq b_n$, $\forall n \in \mathbb{N}$.



$\therefore$ $a_n \leq b_n$, $\forall n \in \mathbb{N}$.

$\therefore$ $a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq b_{n-1} \leq \cdots \leq b_1$

Thus there are two sequences, $\{a_n\}$ and $\{b_n\}$ where $\{a_n\}$ is m.i and $\{b_n\}$ is m.d. Also $\{a_n\}$ is bounded above (by b_1) and $\{b_n\}$ is bouned below (by a_1).

Then by monotone convergence theorem, $\{a_n\}$ and $\{b_n\}$ are both convergent.

Let $\lim a_n = l$ and $\lim b_n = m$

Since $\lim (a_n - b_n) = 0$ (by given condition)

$\therefore$ $\lim a_n = \lim b_n$

ie $l = m = \alpha$ (say) —— (1)

Hence α is the l.u.b of the seqn $\{a_n\}$ and α is the g.l.b of the seqn $\{b_n\}$.

Hence $a_n \leq \alpha$ and $\alpha \leq b_n$, $\forall n \in \mathbb{N}$.

$\therefore$ $a_n \leq \alpha \leq b_n$, $\forall n \in \mathbb{N}$.

$\therefore$ $\alpha \in [a_n, b_n]$, $\forall n \in \mathbb{N}$

ie. $\alpha \in \bigcap_{n=1}^{\infty} [a_n, b_n]$

We now show that α is unique.

Let $\beta (\neq \alpha) \in \bigcap_{n=1}^{\infty} [a_n, b_n]$

Then $a_n \leq \beta \leq b_n$, $\forall n \in N$.

Now, we define a sequence $\{c_n\}$ by

$c_n = \beta$, $\forall n \in \mathbb{N}$

Then $\lim c_n = \beta$. ——— (2)

Now $a_n \leq c_n \leq b_n$, $\forall n \in \mathbb{N}$

and $\lim a_n = \lim b_n = \alpha$ [From (1)]

Then By sandwitcz theorem,

$\lim c_n = \alpha$

ie. $\beta = \alpha$ [by (2)]

Hence α is unique and the theorem is proved.