GE2

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UNIT-1: DIFFERENTIAL CALCULUS-II

Infinite Series of constant terms

Let $\{x_1, x_2, x_3, \ldots\ldots\}$ be a sequence. Then if we consider the sum

$$x_1 + x_2 + x_3 + \cdots + x_n + \cdots\cdots \longrightarrow (1)$$

of all the terms of the sequence, then this sum is called an infinite series and it is denoted by Σx_n.

We now consider a sequence $\{s_n\}$ which is associated with the infinite series Σx_n in the following way:

$$s_1 = x_1$$
$$s_2 = x_1 + x_2$$
$$s_3 = x_1 + x_2 + x_3$$
$$s_4 = x_1 + x_2 + x_3 + x_4$$
$$\cdots\cdots\cdots$$
$$\cdots\cdots\cdots$$
$$s_n = x_1 + x_2 + \cdots\cdots + x_n, \quad \forall n.$$
$$\cdots\cdots\cdots$$

This sequence $\{s_n\}$ is called the sequence of partial sums of the series (1).

If the seq^n $\{s_n\}$ of partial sum converges, then the series ① is convergent.

The limit of the sequence $\{s_n\}$, i.e. $\lim s_n$ is said to be the sum of the series ①.

If the limit of the sequence $\{s_n\}$ does not exist, then the ~~series is also not convergent~~ sum of the series does not exist and hence the series is not convergent.

Result:

An infinite series is said to converge, diverge or ~~oscillate~~ oscillate according as its sequence of partial sums $\{s_n\}$ converges, diverges or oscillates.

Ex:- Let us consider the series

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots$$

Let this series be denoted by Σx_n.

Then the nth term of the series is

$$x_n = \frac{1}{n(n+1)}$$

Now, we consider the sequence of partial sum of the series is $\{s_n\}$, where,

$$s_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$$

$$= \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \cdots + \left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= 1 - \frac{1}{n+1}$$

Now, $\lim_{n\to\infty} s_n = 1$.

So, the sequence $\{s_n\}$ is convergent and limit of the sequence is 1.

Hence the given series is convergent and the sum of the series is 1.

Ex: Let us consider the series $\sum \frac{1}{n}$.

Let the series be $\sum u_n$.

Then $u_n = \frac{1}{n}$.

Let $\{s_n\}$ be the sequence of partial sums of the series $\sum u_n$.

i.e. $s_n = u_1 + u_2 + \cdots + u_n$, $\forall n$.

(4)

Then $s_1 = 1$

$s_2 = 1 + \frac{1}{2}$

$s_3 = 1 + \frac{1}{2} + \frac{1}{3}$

$s_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right)$

$= 1 + 2 \cdot \frac{1}{2}$

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$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$

$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right)$

$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

$= 1 + 3 \cdot \frac{1}{2}$

Similarly, $s_{16} > 1 + 4 \cdot \frac{1}{2}$

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$s_{2^n} > 1 + n \cdot \frac{1}{2}$

$\therefore \lim_{n \to \infty} s_{2^n} = \infty.$

The sequence $\{s_n\}$ is monotonically increasing, as $s_{n+1} - s_n = s_{n+1} > 0, \forall n \in \mathbb{N}.$

As the subsequence $\{s_{2^n}\}$ diverges to ∞ and the sequence $\{s_n\}$ is unbounded, therefore the series is divergent.

* Ex. Show that the series $\Sigma\left(\frac{1}{n}\right)^{1/n}$ diverges.

Sol^n: Here $u_n = \left(\frac{1}{n}\right)^{1/n}$

$$\therefore \log u_n = \left(\frac{1}{n}\right)\log\left(\frac{1}{n}\right)$$
$$= -\frac{1}{n}\log n$$

$$\therefore \lim_{n\to\infty} \log u_n = \lim_{n\to\infty}\left(-\frac{1}{n}\log n\right)$$
$$= -\lim_{n\to\infty}\frac{\log n}{n} \quad \left[\frac{\infty}{\infty} \text{ form}\right]$$
$$= -\lim_{n\to\infty}\frac{\frac{1}{n}}{1} \quad \left[\text{L' Hospital's rule}\right]$$
$$= 0$$

$$\therefore \lim_{n\to\infty} \log u_n = 0$$

or, $\log\left(\lim_{n\to\infty} u_n\right) = 0$

or, $\lim_{n\to\infty} u_n = e^0 = 1 \neq 0$

$\therefore$ The given series is divergent.

Cauchy's principal of Convergence:-

A necessary and sufficient condition for the convergence of a series $\sum u_n$ is that corresponding to a pre-assigned positive ϵ, there exists a natural number m such that

$$|u_{n+1} + u_{n+2} + \cdots + u_{n+p}| < \epsilon, \quad \forall n \geq m,$$

and any natural number p.

Result: A necessary condition for the convergence of a series $\sum u_n$ is $\lim u_n = 0$

But the converse is not true.

Example: Let us consider the series $\sum u_n$, where $u_n = \frac{1}{n}$.

Here $\lim u_n = 0$.

But $\sum u_n$ is a divergent series, for

$$|u_{n+1} + u_{n+2} + \cdots u_{n+p}|$$

$$= \left|\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+p}\right|$$

Now, if we take $p = n$, then

$$\left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right|$$

$$> \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$= \frac{1}{2}$$

$\therefore |S_{n+p} - S_n| \nless \epsilon = \frac{1}{2}$, for every natural no. p.

Hence $\sum \frac{1}{n}$ is divergent.

Series of positive terms

A series $\sum u_n$ is said to be a series of positive terms if every u_n is positive, where n is a natural no.

Result: A series of positive real nos $\sum u_n$ is convergent iff the sequence $\{s_n\}$ of partial sums is bounded ~~above~~ above.

Comparison test

Let $\sum u_n$ and $\sum v_n$ be two series of positive real numbers and there is a natural number m such that $u_n \leq k v_n$, $\forall n \geq m$, k being a fixed positive number.

Then
(i) $\sum u_n$ is convergent if $\sum v_n$ is conv.
(ii) $\sum v_n$ is divergent if $\sum u_n$ is div.

Ex: Examine for convergence the series

$$\sum \frac{1}{n^2+a^2}$$

Here $u_n = \dfrac{1}{n^2+a^2}$, take $v_n = \dfrac{1}{n^2}$

$\therefore u_n \leq v_n, \forall n \geq 1$

Now, $\frac{u_n}{v_n} = \frac{n^2}{n^2+a^2} = \frac{1}{1+\frac{a^2}{n^2}}$

$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{a^2}{n^2}}$

$= 1.$

Since $\sum \frac{1}{n^2}$ is a p-series for $p = 2$, it is convergent, i.e. $\sum v_n$ is convergent. Hence by comparison test $\sum u_n$ is also convergent.

Ex: Test the convergence of the series

$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots + \frac{1}{n^n} + \cdots$

Soln: clearly, $n^n > 2^n$, $n > 2$

$\therefore \frac{1}{n^n} < \frac{1}{2^n}$

Now, $\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$

is a geometric series with common ratio $\frac{1}{2} < 1$. So $\sum \frac{1}{2^n}$ is convergent.

Then by first comparison test $\sum \frac{1}{n^n}$ is convergent.