TOPIC : FOURIER SERIES

SUBJECT: MATHEMATICS

SEMESTER: 4, CORE COURSE: 8

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1.1. Introduction :

In many engineering problems, especially in the study of periodic phenomenae e.g in conduction of heat, electro-dynamics etc. the periodic functions occur frequently. These functions can be expressed as a series of sines and cosines which is known as Fourier Series. Expression into this series is used to solve several difficult problems in technology since 1700 A.D. Though the rigorous theory of Fourier Series is complicated its application is so simple. Its field of application is more wide than that of Taylor's/ Macluirin's series because many discontinuous function can be expressed into a Fourier Series.

In this chapter we first introduce perodic functions and then develope its Fourier series.

1.2. Some Special Functions.

In this article we introduce the following functions which are very well known to the readers.

Periodic Functions and its Properties : A function $f(x)$ is said to be periodic if there exists a positive number T such that $f(x+T)=f(x)$ for all values of x. For example $f(x)=\sin x$ is a periodic function of period 2π because $f(x+2\pi)=\sin(x+2\pi)=\sin x=f(x)$.

Note that a periodic function is defined on $(-\infty,\infty)$.

(i) Sum or difference of two periodic functions with common period T is also periodic with period T.

(ii) If $f(x)$ is periodic with period T then $f(ax)$ is periodic with period $\frac{T}{a}$ e.g $\sin 3x$ is periodic with period $\frac{2\pi}{3}$.

(iii) Any function $f(x)$ defined on a finite interval $[a,b]$ can be extended to a 'periodic function $f(x)$ of period $T=b-a$' directly by defining $f(x+T)=f(x)$ for all x. This can be illustrated graphically. Let the adjacent Fig. 1 shows the graph of $f(x)$.

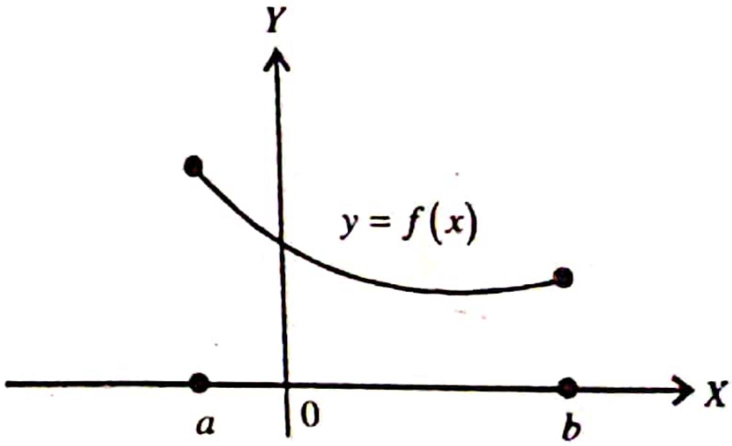


Fig. 1

After extension to a periodic function of period $T=b-a$ we have the following graph (Fig. 2)

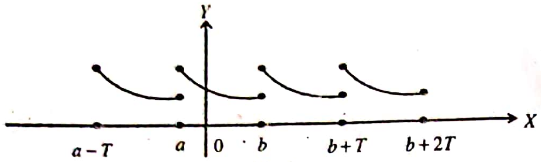


Fig. 2

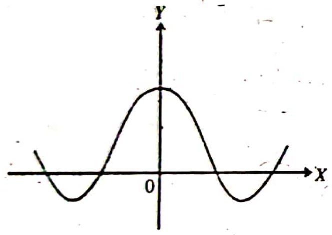
This extended function is defined everywhere.

Even and Odd Function; Their Properties

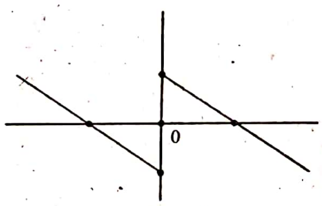
A funtion $f(x)$ is said to be even funtion if $f(-x)=f(x)$ for all values of x; e.g the functions $\cos x$, x^2 all are even functions.

A function $f(x)$ is said to be odd function if $f(-x)=-f(x)$ for all values of x. e.g. the functions $\sin x$, x^3 etc are all odd functions.

(i) The graph of these functions shown in the following figures :



Graph of an even function



Graph of an odd function

(ii) One of the most important properties of even function is $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$.

(iii) One of the most important properties of odd function is $\int_{-a}^{a} f(x)dx = 0$.

1.3. Typical Waveform

The graph of every periodic function runs like a wave — this is wave-form. Below we show some typical wave–form which are usually met in communication engineering :

(i) *Square Waveform* : Consider the periodic function $f(x)$ defined by

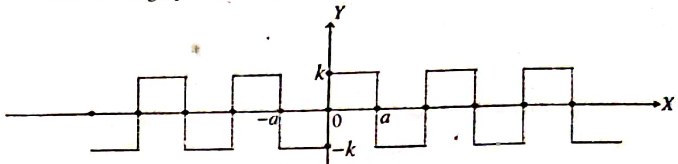
$f(x) = -k$, $-a < x < 0$

$= k$, $0 < x \le a$

and $f(x+2a) = f(x)$ for all x.

Graph of this periodic function is shown below.

This kind of graph is known as Square Waveform.

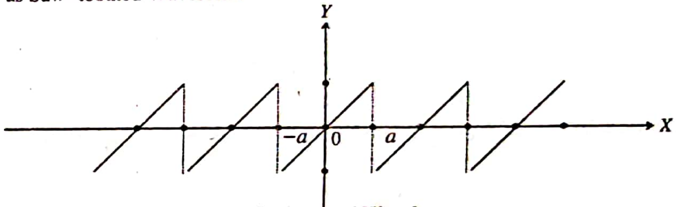


Square Waveform

(ii) *Saw – toothed Waveform* :

Consider the periodic function $f(x)$ defined by $f(x) = x$, $-a < x \le a$ and $f(x+2a) = f(x)$ for all x.

Graph of this periodic function is shown below. This kind of graph is known as Saw -toothed Waveform.



Saw-toothed Waveform

(iii) *Triangular Waveform* :

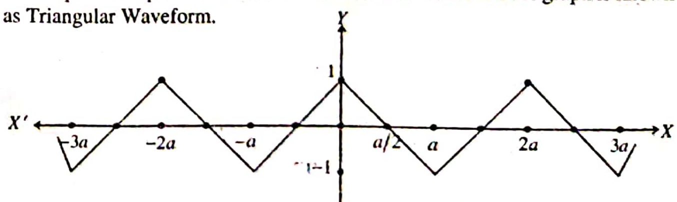
Consider the periodic functions $f(x)$ defined by

$$f(x) = 1 + \frac{2x}{a}, \quad -a < x \le 0$$

$$= 1 - \frac{2x}{a}, \quad 0 \le x \le a$$

and $f(x+2a) = f(x)$ for all x.

Graph of this periodic function is shown below. This kind of graph is known as Triangular Waveform.



Triangular Wave-form

Note : (1) There has no gurantee that every function $f(x)$ equals its Fourier Series. In the subsequent article we shall discuss the conditions under which this equality would hold good.

(2) Since T can assume any value so this Fourier Series can be regarded as general Fourier Series.

Illustration

Consider the function $f(x)=3, 0<x\le 5$

$= -3, -5<x\le 0$

We extend the function by defining $f(x+10)=f(x)$ for all x. So this becomes a periodic function of period 10. This gives a square waveform.

Its Fourier co-efficients, according to Euler Formula, are

$$a_0=\frac{1}{5}\int_{-5}^{5} f(x)dx$$

$$=\frac{1}{5}\left\{-3\int_{-5}^{0} dx+3\int_{0}^{5} dx\right\} \quad =0$$

$$a_n=\frac{1}{5}\int_{-5}^{5} f(x)\cos\frac{n\pi x}{5}dx$$

$$=\frac{1}{5}\left\{-3\int_{-5}^{0}\cos\frac{n\pi x}{5}dx+3\int_{0}^{5}\cos\frac{n\pi x}{5}dx\right\}$$

$$=\frac{1}{5}\left\{-3\int_{0}^{5}\cos\frac{n\pi x}{5}dx+3\int_{0}^{5}\cos\frac{n\pi x}{5}dx\right\}=0$$

and $b_n=\frac{1}{5}\int_{-5}^{5} f(x)\sin\frac{n\pi x}{5}dx$

$$=\frac{2}{5}\int_{0}^{5} f(x)\sin\frac{n\pi x}{5}dx \quad [\because f(x)\sin\frac{n\pi x}{5} \text{ is even function}]$$

$$=\frac{6}{5}\int_{0}^{5}\sin\left(\frac{n\pi x}{5}\right)dx=\frac{-6}{5}\left[\frac{\cos\frac{n\pi x}{5}}{\frac{n\pi}{5}}\right]_0^5$$

$$=-\frac{6}{n\pi}(\cos n\pi-1)=\frac{6(1-\cos n\pi)}{n\pi}$$

Therefore the Fourier series of $f(x)$ is

$$\frac{0}{2}+\sum_{n=1}^{\infty}\left(0\cdot\cos\frac{n\pi x}{5}+\frac{6(1-\cos n\pi)}{n\pi}\sin\frac{n\pi x}{5}\right)$$

i.e. $\sum_{n=1}^{\infty}\frac{6}{\pi}\cdot\frac{1-\cos n\pi}{n}\sin\frac{n\pi x}{5}$

i.e. $\frac{6}{\pi}\left\{(1-\cos\pi)\sin\frac{\pi x}{5}+\frac{(1-\cos 2\pi)}{2}\sin\frac{2\pi x}{5}+\frac{(1-\cos 3\pi)}{3}\sin\frac{3\pi x}{5}+\cdots\cdots\right\}$

We see $f(0)=-3$ but the values of the Fourier Series at $x=0$

is $\frac{6}{\pi}\{0+0+0+\cdots\cdots\}=0$.

Fourier Series of a Function of Period 2π

The above Fourier Series for $T=\pi$ i.e. the Fourier series for the function $f(x)$ defined and integrable on $(-\pi,\pi)$ and $f(x+2\pi)=f(x)$ for all values of x, is

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

where the Fourier Co-efficients are

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

and $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$, for $n=1,2,3\cdots\cdots$

Illustration

Consider the function, $f(x)=x^2, -\pi<x\le\pi$

This function is defined on the interval $(-\pi,\pi]$.

We extend this by defining $f(x+2\pi)=f(x)$ for all values of x. This is a periodic function of period 2π. Its Fourier co-efficients are

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2dx=\frac{2}{\pi}\int_{0}^{\pi} x^2dx=\frac{2\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos nx\,dx=\frac{2}{\pi}\int_{0}^{\pi} x^2\cos nx\,dx \quad [\because x^2\cos nx \text{ is even function}]$$

$$= \frac{2}{\pi}\left\{\left[x^2 \frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi} 2x \frac{\sin nx}{n} dx\right\}$$

$$= -\frac{4}{\pi n}\left\{\left[-x\frac{\cos nx}{n}\right]_0^{\pi} + \int_0^{\pi} \frac{\cos nx}{n} dx\right\}$$

$$= -\frac{4}{\pi n}\left\{-\frac{\pi \cos n\pi}{n} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right\} = \frac{4\cos n\pi}{n^2}$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \sin nx\, dx = 0$ [$\because x^2 \sin nx$ is odd function]

So, the Fourier series of $f(x)$ is

$$\frac{1}{2}\cdot\frac{2\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4\cos n\pi}{n^2}\cos nx + 0\cdot \sin nx\right)$$

i.e. $\frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{\cos n\pi \cos nx}{n^2}$.

1.5. Fourier Expansion and its Conditions

We have earlier remarked that the Fourier Series of a function $f(x)$ may not be equal to $f(x)$. We also gave an example in this regard.

However when the Fourier series of $f(x)$ becomes equal to $f(x)$ we say the series as a Fourier Expansion of $f(x)$.

Next we are going to state the condition under which a function will have Fourier Expansion.

Dirichlet's Conditions.

A function $f(x)$ defined on $[-T,T]$ is said to satisfy **Dirichlet's** Conditions if it satisfies any one of the following two conditions :

(1) $f(x)$ is bounded in $[-T,T]$ and the interval $[-T,T]$ can be decomposed into a finite number of sub-intervals such that $f(x)$ is monotonic (increasing or decreasing) on each of the sub-intervals.

(2) $f(x)$ has a finite number of points of infinite discontinuity in $[-T,T]$.

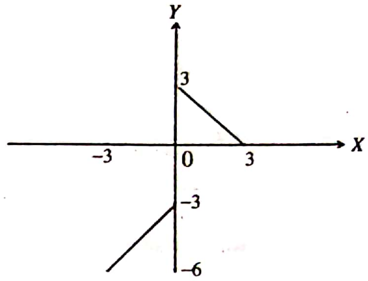
When arbitrary small neighbourhood of these points are excluded from $[-T,T]$ $f(x)$ becomes bounded in the remaining part and this remaining part can be decomposed into a finite number of sub-intervals such that $f(x)$ is monotonic in each of the sub-intervals. Moreover the improper integral $\int_{-\pi}^{\pi} f(x)dx$ is absolutely convergent.

Illustrations.

(i) Let $f(x) = x-3,\ -3 \le x \le 0$

$= 3-x,\ 0 < x \le 3.$

Let us draw the graph of $f(x)$ below :

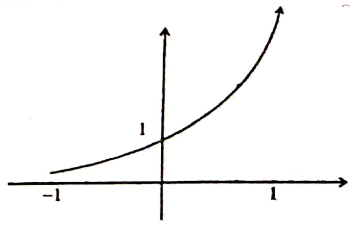


From the graph we see $-6 \le f(x) \le 3$ i.e. $f(x)$ is bounded in $[-3,3]$.

The interval $[-3,3]$ is decomposed as $[-3,3] = [-3,0] \cup [0,3]$ such that $f(x)$ is increasing in $[-3,0]$ and decreasing in $[0,3]$. So we conclude this function $f(x)$ satisfies Dirichlet's Condition.

(ii) Let $f(x) = \frac{1}{\sqrt{1-x}},\ -1 \le x \le 1.$

Its graph is shown below :



Though this function is not bounded in $[-1,1]$ it has one (finite) point of infinite discontinuity (at $x=1$). If a neighbourhood of $x=1$ is excluded the functions becomes bounded and $f(x)$ is monotonic increasing in the remaining part.

Moreover the improper integral $\int_{-1}^{1}\frac{dx}{\sqrt{1-x}}$ is absolutely convergent.

So this function satisfies Dirichlet's Condition.

Theorem

If a function $f(x)$ defined on $[-T,T]$ satisfies Dirichlet's Condition then its Fourier series is convergent and the Fourier series

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{T}+b_n\sin\frac{n\pi x}{T}\right)=f(x)\text{, if } x \text{ is the point of continuity of } f(x)$$

$$=\frac{1}{2}\left\{\lim_{t\to x+}f(t)+\lim_{t\to x-}f(t)\right\}\text{, if } x \text{ is a point of ordinary discontinuity}$$

$$=\frac{1}{2}\left\{\lim_{t\to -T+}f(t)+\lim_{t\to T-}f(t)\right\}\text{ at } x=\pm T$$

Proof. Beyond the scope of the book.

Illustration

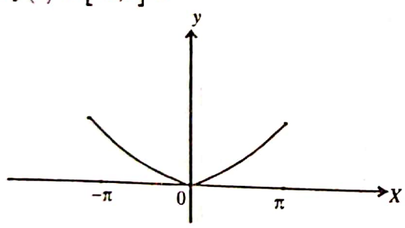
Again consider the function

$f(x)=x^2,\ -\pi<x\le\pi$

Extending this to periodic by defining $f(x+2\pi)=f(x)$ for all values of x we got its Fourier Series as (see a previous illustration)

$$\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos n\pi\cos nx}{n^2}$$

The graph of $f(x)$ in $[-\pi,\pi]$ is



From the graph it is clear that $f(x)$ is bounded and monotonic in two sub intervals $[-\pi,0]$ and $[0,\pi]$. So $f(x)$ satisfies Dirichlet's Condition. $f(x)$ is continuous every where in $(-\pi,\pi)$.

Now, $\frac{1}{2}\left\{\lim_{t\to-\pi+}f(t)+\lim_{t\to\pi-}f(t)\right\}$

$$=\frac{1}{2}\left\{\lim_{t\to-\pi}t^2+\lim_{t\to\pi}t^2\right\}=\frac{1}{2}\left(\pi^2+\pi^2\right)=\pi^2=f(\pi)$$

So, the Fourier Expansion of the given function is

$$f(x)=\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{\cos n\pi\cos nx}{n^2}.$$

1.6 Change of Interval

We introduce Fourier Series of a function $f(x)$ which is primarily defined on the interval $[-T,T]$ and then extending it to a periodic wave.

But in many engineering problem the function may appear as defined primarily on an interval $[c,c+2T]$ where c may be any real number. In that case also we have no trouble of getting its Fourier Series. In fact the following theorem assures so.

Theorem

If $f(x)$ be defined and integrable in $[c,c+2T]$ and $f(x+2T)=f(x)$ for all values of x (i.e. $f(x)$ periodic of period $2T$) then the Fourier Series of $f(x)$ is also

$$\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{T}+b_n\sin\frac{n\pi x}{T}\right)$$

where the Fourier co-efficients are

$$a_0=\frac{1}{T}\int_c^{c+2T}f(x)dx$$

$$a_n=\frac{1}{T}\int_c^{c+2T}f(x)\cos\frac{n\pi x}{T}dx$$

$$b_n=\frac{1}{T}\int_c^{c+2T}f(x)\sin\frac{n\pi x}{T}dx$$

where c may be any real number.

Proof. Omitted.

Illustration.

Consider the function $f(x)=x^2,\ 0<x\le2\pi$.

Here the function is defined in $[0,2\pi]$. Here $c=0$, $c+2T=2\pi$ $\therefore T=\pi$.

$\therefore$ the Fourier co-efficients are

$$a_0=\frac{1}{\pi}\int_0^{0+2\pi}x^2dx=\frac{8\pi^2}{3}$$

$$a_n=\frac{1}{\pi}\int_0^{0+2\pi}x^2\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}x^2\cos nx\,dx=\frac{4}{n^2}$$

(can be had by integ by parts)

$$b_n = \frac{1}{\pi}\int_0^{0+2\pi} x^2 \sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx = -\frac{4\pi}{n}$$

(evaluations are not shown).

So the Fourier Series of the given function (after extending to a periodic function of period 2π) is

$$\frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right).$$

Clearly $f(x)$ satisfies Dirichlet's Condition. So the Fourier expansion is

$$\frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx - \frac{4\pi}{n}\sin nx\right) = x^2,\ 0 < x \le 2\pi .$$

1.7. Half Range Series : Sine or Cosine

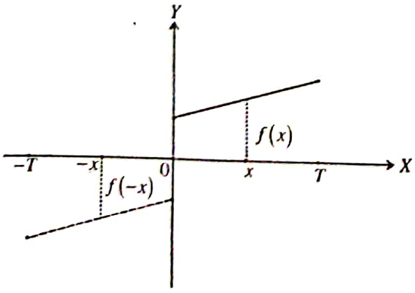
A trigonometric series like the Fourier Series is called a Half Range Series if only sine terms or only cosine terms are present.

When only sine terms are present the series is called *Half Range Sine series*; when only cosine terms are present the series is called *Half Range Cosine Series.*

When a half range series corresponding to a function is desired, the function is generally defined in the interval $(0,T)$ which is half of the interval $(-T,T)$.

Construction of Half Range Sine Series.

Let $f(x)$ be a function defined and integrable on the interval $(0,T)$. We extend the domain of definition to $[-T,0]$ defining by $f(-x) = -f(x)$. This extension is shown in the adjacent figure. Then this extended $f(x)$ becomes odd in the interval $[-T,T]$.



Therefore, $a_0 = \frac{1}{T}\int_{-T}^{T} f(x)dx$

$= 0 \qquad \because f(x)$ is odd.

$$a_n = \frac{1}{T}\int_{-T}^{T} f(x)\cos\frac{n\pi x}{T}dx$$

$= 0$ [$\because f(x)\cos\frac{n\pi x}{T}$ is odd function]

and $b_n = \frac{1}{T}\int_{-T}^{T} f(x)\sin\frac{n\pi x}{T}dx$

$= \frac{2}{T}\int_0^{T} f(x)\sin\frac{n\pi x}{T}dx$ [$\because f(x)\sin\frac{n\pi x}{T}$ is even function]

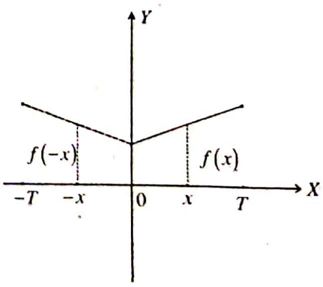
The Fourier series of $f(x)$ becomes

$$\frac{0}{2} + \sum_{n=1}^{\infty}\left(0\cos\frac{n\pi x}{T} + b_n \sin\frac{n\pi x}{T}\right) \text{ i.e. } \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{T}$$

which is the required Half Range Sine Series. Obviously if $f(x)$ satisfies Dirichlets condition in $[0,T]$ then this series is convergent and the value is as for (Full Range) Fourier Series.

Construction of Half Range Cosine Series.

Let $f(x)$ be a function defined and integrable on the interval $(0,T)$. We extend the domain of definition to $[-T,0]$ defining by $f(-x) = f(x)$. This extension is shown in the adjacent figure. Then this extended $f(x)$ becomes an even function in the interval $[-T,T]$.



1.8. Parseval's Identity

If the Fourier Series of a function $f(x)$ converges uniformly to $f(x)$ in the interval $(-T,T)$ then

$$\frac{1}{T}\int_{-T}^{T}\{f(x)\}^2\,dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right) \qquad (1)$$

where a_n, b_n are Fourier Co-efficients of $f(x)$.

Proof. Beyond the scope of the book.

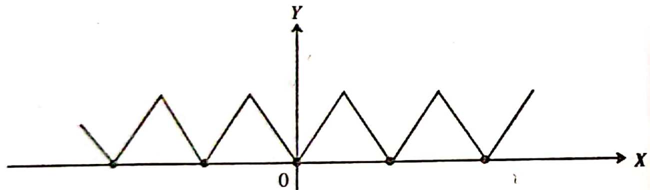
Note. The relation (1) is Parseval's identity. This is valid under less restrictive conditions than that imposed in the above theorem.

Illustration.

Consider the function $f(x) = -x$, $-2 < x \le 0$

$= x$, $0 \le x \le 2$.

We see $f(x)$ is an even function. Extending this to a periodic function defining by $f(x+4) = f(x)$ we get the graph as follow :



Here $a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\int_{-2}^{0} -x\,dx + \frac{1}{2}\int_{0}^{2} x\,dx = 2$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{2}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}dx$$

$[\because f(x)\cos\frac{n\pi x}{2}$ is an even function$]$

$$= \left[x\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - 1\left(\frac{-4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^2$$

$$= \frac{4}{n^2\pi^2}(\cos n\pi - 1) \text{ for } n \ne 0.$$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = 0 \quad [\because f(x)\sin\frac{n\pi x}{2} \text{ is an odd function}]$$

Again the function $f(x)$ satisfies Dirichlet's Condition and it is continuous everywhere (which is seen from its graph).

So its Parseval's Identity is

$$\frac{1}{2}\int_{-2}^{2}\{f(x)\}^2 dx = \frac{2^2}{2} + \sum_{n=1}^{\infty}\left[\left\{\frac{4}{n^2\pi^2}(\cos n\pi - 1)\right\}^2 + 0^2\right]$$

or, $\frac{1}{2}\left[\int_{-2}^{0}(-x)^2 dx + \int_{0}^{2} x^2 dx\right] = 2 + \sum_{n=1}^{\infty}\frac{16}{n^4\pi^4}(\cos n\pi - 1)^2$

or, $\frac{1}{2}\left[\frac{x^3}{3}\right]_{-2}^{2} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\cdots\right)$

or, $\frac{8}{3} = 2 + \frac{64}{\pi^4}\left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\cdots\right)$

or, $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\cdots = \frac{\pi^4}{96}$, an interesting result is obtained.

Illustrative Examples.

Ex. 1. *Find a Fourier series of the function* $f(x) = x - x^2$, $-\pi < x \le \pi$. *Hence find the value of the series*

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$$

The function is defined on $(-\pi,\pi]$ primarily. We extend by defining outside as $f(x+2\pi) = f(x)$. It becomes a periodic function of period 2π.

The Fourier Co-efficients are

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)dx = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(x - x^2\right)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[\left(x - x^2\right)\frac{\sin nx}{n}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}(1-2x)\frac{\sin nx}{n}dx\right\}$$

$$= \frac{-1}{n\pi}\int_{-\pi}^{\pi}(1-2x)\sin nx\,dx$$

$$=\frac{-1}{n\pi}\left\{\left[-(1-2x)\frac{\cos nx}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(-2)\frac{(-)\cos nx}{n}dx\right\}$$

$$=\frac{1}{n\pi}\left\{\frac{(1-2\pi)\cos n\pi}{n}-\frac{(1+2\pi)\cos n\pi}{n}+\frac{2}{n}\int_{-\pi}^{\pi}\cos nx\,dx\right\}$$

$$=\frac{4(-1)^{n+1}}{n^2} \text{ for } n\neq 0.$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\left(x-x^2\right)\sin nx\,dx$$

$$=\frac{1}{\pi}\left\{\left[\frac{-\left(x-x^2\right)\cos nx}{n}\right]_{-\pi}^{\pi}-\int_{-\pi}^{\pi}(1-2x)\frac{-\cos nx}{n}dx\right\}$$

$$=\frac{-1}{\pi}\left\{\frac{\left(\pi-\pi^2\right)\cos n\pi}{n}-\frac{\left(-\pi-\pi^2\right)\cos n\pi}{n}-\frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\,dx\right\}$$

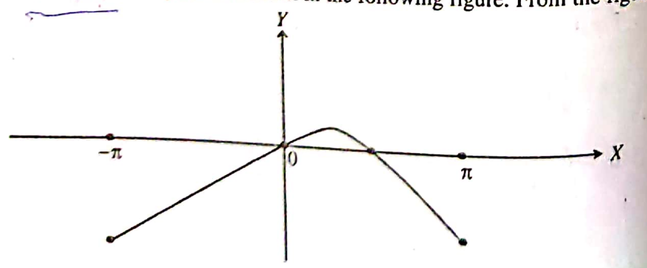
$$=\frac{2(-1)^{n+1}}{n}$$

So the Fourier series of the given function is

$$\frac{1}{2}\left(-\frac{2\pi^2}{3}\right)+\sum_{n=1}^{\infty}\left\{\frac{4(-1)^{n+1}}{n^2}\cos nx+\frac{2\cdot(-1)^{n+1}}{n}\sin nx\right\}$$

$$=-\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Now, $y=x-x^2$ is a parabola shown in the following figure. From the figure



we see it is bounded and monotonic in two subintervals. So $f(x)$ satisfies Dirichlet's condition. Since the function is continuous in $(-\pi,\pi)$ so

$$x-x^2=-\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos nx+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

Putting $x=0$ we get

$$0=-\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}+2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\times 0$$

or, $$\frac{\pi^2}{3}=4\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\cdots\right)$$

or, $$\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots\cdots=\frac{\pi^2}{12}.$$

Ex. 2. *Write the Fourier Series of the function*

$$f(x)=0,\ -5<x<0$$
$$=3,\ 0<x<5$$

and $f(x)$ is periodic function of period 10. How should $f(x)$ be defined at $x=-5$, $x=0$ and $x=5$ so that its Fourier series converges to $f(x)$ for $-5\le x\le 5$.

The function is defined primarily on $(-5,5)$. Therefore the Fourier Co-efficients

$$a_0=\frac{1}{5}\int_{-5}^{5}f(x)dx \qquad =\frac{1}{5}\left\{\int_{-5}^{0}o\,dx+\int_{0}^{5}3dx\right\}=3$$

For $n\neq 0$ $a_n=\frac{1}{5}\int_{-5}^{5}f(x)\cos\frac{n\pi x}{5}dx$

$$=\frac{1}{5}\int_0^5 3\cos\frac{n\pi x}{5}dx \text{ [by def. of } f(x)\text{]}$$

$$=\frac{3}{5}\left[\frac{5}{n\pi}\sin\frac{n\pi x}{5}\right]_0^5=\frac{3}{n\pi}\{\sin n\pi-0\}=0$$

and $b_n=\frac{1}{5}\int_{-5}^{5}f(x)\sin\frac{n\pi x}{5}dx=\frac{1}{5}\int_0^5 3\sin\frac{n\pi x}{5}dx$

$$=\frac{3}{5}\left[\frac{-5}{n\pi}\cos\frac{n\pi x}{5}\right]_0^5=-\frac{3}{n\pi}\left[\cos\frac{n\pi x}{5}\right]_0^5$$

$$=-\frac{3}{n\pi}\{\cos n\pi-\cos 0\}=\frac{3}{n\pi}(1-\cos n\pi)\quad=\frac{3}{n\pi}\left(1-(-1)^n\right).$$

So the Fourier Series is

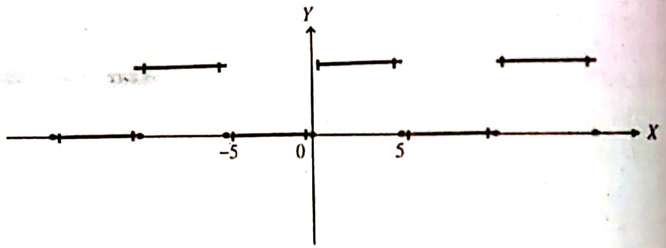
$$\frac{3}{2}+\sum_{n=1}^{\infty}\left\{0\cdot\cos\frac{n\pi x}{5}+\frac{3}{n\pi}\left(1-(-1)^n\right)\sin\frac{n\pi x}{5}\right\}$$

$$=\frac{3}{2}+\frac{3}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\sin\frac{n\pi x}{5}$$

$$=\frac{3}{2}+\frac{3}{\pi}\left\{2\sin\frac{\pi x}{5}+\frac{2}{3}\sin\frac{3\pi x}{5}+\frac{2}{5}\sin\frac{5\pi x}{5}+\cdots\cdots\right\}$$

$$=\frac{3}{2}+\frac{6}{\pi}\left\{\sin\frac{\pi x}{5}+\frac{1}{3}\sin\frac{3\pi x}{5}+\frac{1}{5}\sin\frac{5\pi x}{5}+\cdots\cdots\right\}$$

We draw the graph of the function below :



From the graph we see $f(x)$ is bounded in $[-5,5]$ and monotonic in the two sub-intervals $(-5,0)$ and $(0,5)$.

So $f(x)$ satisfies Dirichlet's Condition in $(-5,5)$. It is discontinuous at $x=0$. Therefore at $x=0$ the Fourier series converges to

$$\frac{1}{2}\left\{\lim_{t\to 0+}f(t)+\lim_{t\to 0-}f(t)\right\}=\frac{1}{2}\left\{\lim_{t\to 0+}3+\lim_{t\to 0-}0\right\}=\frac{3}{2}.$$

So for convergence of $f(x)$ at $x=0$ we should define $f(0)=\frac{3}{2}$.

At the end points $x=-5$ and $x=5$ the Fourier Series converges to

$$\frac{1}{2}\left\{\lim_{t\to -5+}f(t)+\lim_{t\to 5-}f(t)\right\}$$

$$=\frac{1}{2}\left\{\lim_{t\to -5+}0+\lim_{t\to 5-}3\right\}=\frac{3}{2}.$$

So for convergence of $f(x)$ at $x=-5$ and at $x=5$ we should define $f(-5)=\frac{3}{2}$, $f(5)=\frac{3}{2}$.

Ex. 3. *Obtain the Fourier expansion of $x\sin x$ is $-\pi\le x\le\pi$ and deduce that*

$$\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots\cdots$$

$f(x)=x\sin x$ satisfies Dirichlet's condition in $(-\pi,\pi)$ and it is continuous everywhere so we can say it can be expanded into its Fourier series, i. e.

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx).$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx\qquad=\frac{2}{\pi}\int_0^{\pi}x\sin x\,dx\qquad=2$$

and for $n\neq 0$ $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin x\cos nx\,dx$

$$=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos nx\,dx\quad[\because x\sin x\cos nx \text{ is even function}]$$

$$=\frac{1}{\pi}\int_0^{\pi}x\{\sin(n+1)x+\sin(1-n)x\}dx$$

$$=\frac{1}{\pi}\left[x\left(\frac{-\cos(n+1)x}{n+1}-\frac{\cos(1-n)x}{1-n}\right)\right]_0^{\pi}-\int_0^{\pi}\left\{\frac{-\cos(n+1)x}{n+1}-\frac{\cos(1-n)x}{1-n}\right\}dx$$

$$=\frac{1}{\pi}\left[-\pi\left\{\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(1-n)\pi}{1-n}\right\}\right]+\frac{1}{\pi}\int_0^{\pi}\left(\frac{\cos(n+1)x}{n+1}+\frac{\cos(1-n)x}{1-n}\right)dx$$

$$=-\left\{\frac{\cos(1-n)\pi}{1-n}+\frac{\cos(n+1)\pi}{n+1}\right\}+\frac{1}{\pi}\left[\frac{\sin(n+1)x}{(n+1)^2}+\frac{\sin(1-n)x}{(1-n)^2}\right]_0^{\pi}$$

$$=-\left\{\frac{(-1)^{1-n}}{1-n}+\frac{(-1)^{n+1}}{1+n}\right\}=-\left\{\frac{(-1)^{n+1}}{1-n}+\frac{(-1)^{n+1}}{1+n}\right\}\quad\left[\because(-1)^{1-n}=(-1)^{n+1}\right]\ ?$$

$$= -(-1)^{n+1}\left\{\frac{1}{1-n}+\frac{1}{1+n}\right\} = (-1)^n \frac{1+n+1-n}{1-n^2}$$

$$= \frac{2\cdot(-1)^n}{\left(1-n^2\right)} \text{ for } n \neq 1.$$

Now, $a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx = \frac{1}{\pi}\left\{\left[\frac{-x\cos 2x}{2}\right]_0^{\pi} + \int_0^{\pi}\frac{\cos 2x}{2}dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{\pi}{2}+\frac{1}{2}\int_0^{\pi}\cos 2x\,dx\right\} = \frac{1}{\pi}\left\{-\frac{\pi}{2}+\frac{1}{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{-\frac{\pi}{2}+\frac{1}{2}\times 0\right\} = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\,\sin nx\,dx$$

$= 0$ $\quad$ [$\because x\sin x\sin nx$ is odd function]

$$\therefore x\sin x = \frac{1}{2}\cdot 2 + (a_1\cos x + b_1\sin x) + \sum_{n=2}^{\infty}\left\{\frac{2(-1)^n}{\left(1-n^2\right)}\cos nx + 0\cdot\sin nx\right\}$$

or, $x\sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty}\frac{2(-1)^n}{(1+n)(1-n)}\cos nx$

$$= 1 - \frac{1}{2}\cos x + 2\left\{\frac{1}{3\cdot(-1)}\cos 2x + \frac{-1}{4(-2)}\cos 3x + \frac{1}{5(-3)}\cos 4x + \cdots\cdots\right\}$$

or, $x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{1.3} - \frac{\cos 3x}{2.4} + \frac{\cos 4x}{3.5} - \cdots\cdots\right\}$.

Putting $x = \frac{\pi}{2}$ we get

$$\frac{\pi}{2}\cdot\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} - 2\left\{\frac{\cos\pi}{1.3} - \frac{\cos 3\cdot\frac{\pi}{2}}{2.4} + \frac{\cos 2\pi}{3.5} - \cdots\cdots\right\}$$

or, $\frac{\pi}{2} = 1 - 2\left\{-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\cdots\right\}$

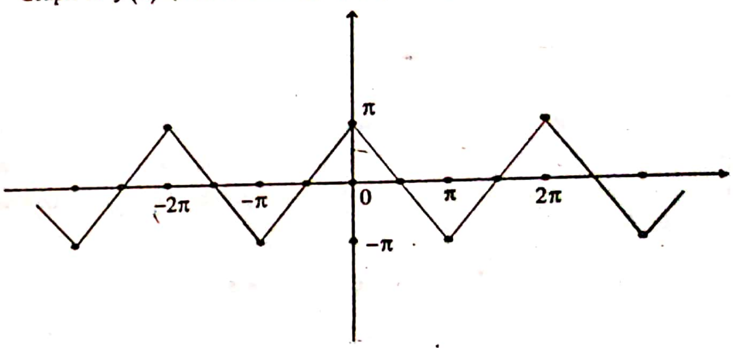
or, $\frac{\pi}{4} = \frac{1}{2} - \left\{-\frac{1}{1.3} + \frac{1}{3.5} - \frac{1}{5.7} + \cdots\cdots\right\}$ $\quad$ or, $\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots\cdots$

Ex. 4. *Draw the graph of the function and state the Wave-form :*

$$f(x) = \pi + 2x,\ -\pi < x < 0$$
$$= \pi - 2x,\ 0 \le x \le \pi.$$

Deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\cdots = \frac{\pi^2}{8}$

Graph of $f(x)$ (after extending by $f(x+2\pi) = f(x)$) is



It gives Triangular Wave form.

From the graph we see $f(x)$ is even function.

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx$ $\because f(x)$ is even

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)dx = \frac{2}{\pi}\left[\pi x - x^2\right]_0^{\pi} = \frac{2}{\pi}\left(\pi^2 - \pi^2\right) = 0.$$

Now, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

[$\because f(x)\cos nx$ is even]

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\cos nx\,dx = \frac{2}{\pi}\left\{\pi\int_0^{\pi}\cos nx\,dx - 2\int_0^{\pi} x\cos nx\,dx\right\}$$

$$= \frac{2}{\pi}\left[\pi\left[\frac{\sin nx}{n}\right]_0^{\pi} - 2\left\{\left[x\frac{\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}dx\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n}\int_0^{\pi}\sin nx\,dx\right] = \frac{-4}{\pi n}\left[\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{4}{\pi n^2}(\cos n\pi - 1)$$

$$= -\frac{4}{\pi n^2}\left((-1)^n - 1\right) = \frac{4}{\pi n^2}\left(1 - (-1)^n\right) \text{ for } n \neq 0$$

For $n>0$, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=0$ [$\because f(x)\sin nx$ is odd function]

$\therefore$ the Fourier series of $f(x)$ is

$$\frac{0}{2}+\sum_{n=1}^{\infty}\frac{4}{\pi n^2}\left(1-(-1)^n\right)\cos nx$$

$$=\frac{4}{\pi}\left(\frac{2}{1^2}\cos x+\frac{2}{3^2}\cos 3x+\frac{2}{5^2}\cos 5x+\cdots\cdots\right)$$

$$=\frac{8}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots\cdots\right)$$

From the graph we see $f(x)$ is bounded in the interval $[-\pi,\pi]$ and it is monotonic in the two sub-intervals $(-\pi,0)$ and $(0,\pi)$. So $f(x)$ satisfies Dirichlet's Condition. Moreover we see, from the graph, $f(x)$ is continuous everywhere. So

$$f(x)=\frac{8}{\pi}\left(\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\frac{1}{5^2}\cos 5x+\cdots\cdots\right)$$

Putting $x=0$ we get, $f(0)=\frac{8}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right)$

or, $\pi=\frac{8}{\pi}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right)$ or, $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$.

Ex. 5. *Find a series of sines and cosines of multiples of x which represents $f(x)$ in the interval $-\pi<x<\pi$ when*

$$f(x)=0,\ -\pi<x\le 0$$
$$=\frac{\pi x}{4},\ 0<x<\pi.$$

Hence deduce that $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$.

The Fourier co-efficients,

$$a_0=\frac{1}{\pi}\int_0^{\pi}\frac{\pi x}{4}dx=\frac{\pi^2}{8}$$

and $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}\frac{\pi x}{4}\cos nx\,dx=\frac{1}{4n^2}(\cos n\pi-1)$ for $n\neq 0$.

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=\frac{1}{\pi}\int_0^{\pi}\frac{\pi x}{4}\sin nx\,dx=-\frac{\pi}{4n}\cos n\pi.$$

Obviously $f(x)$ satisfies Dirichlet's condition. So, as $f(x)$ is continuous

$-\pi<x<\pi$, $f(x)=\frac{1}{2}\cdot\frac{\pi^2}{8}+\sum_{n=1}^{\infty}\left\{\frac{1}{4n^2}(\cos n\pi-1)\cos nx-\frac{\pi}{4n}\cos n\pi\sin nx\right\}$

or, $f(x)=\frac{\pi^2}{16}+\frac{1}{2}\left(-\cos x+\frac{\pi}{2}\sin x\right)-\frac{\pi}{8}\sin 2x+\cdots\cdots$

when $-\pi<x<\pi$.

At $x=\pi$, the Fourier series converges to

$$\frac{1}{2}\left\{\lim_{t\to-\pi+}f(t)+\lim_{t\to\pi-}f(t)\right\}=\frac{1}{2}\left(0+\lim_{t\to\pi}\frac{\pi t}{4}\right)=\frac{\pi^2}{8}.$$

$\therefore$ Putting $x=\pi$ in the F-series we get

$$\frac{\pi^2}{8}=\frac{\pi^2}{16}+\frac{1}{2}\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\right)$$

or, $1+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots=\frac{\pi^2}{8}$.

Ex. 6. *Expand $\cos px$ in $[-\pi,\pi]$ (p not being an integer) in Fourier series. Ded... the value of* $\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+p}+\frac{1}{n+1-p}\right)$

Here $f(x)=\cos px$ is an even function. Its Fourier Co-efficients are

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos px\,dx=\frac{1}{p\pi}[\sin px]_{-\pi}^{\pi}=\frac{2}{p\pi}\sin p\pi$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}\cos px\cos nx\,dx$$

$=\frac{2}{\pi}\int_0^{\pi}\cos px\cos nx\,dx$ [$\because \cos px\cos nx$ is even function]

$$=\frac{1}{\pi}\int_0^{\pi}\{\cos(p+n)x+\cos(p-n)x\}dx$$

$$=\frac{\sin(p+n)\pi}{(p+n)\pi}+\frac{\sin(p-n)\pi}{(p-n)\pi}.$$

Now $\sin(p+n)\pi = \sin(p-n)\pi = (-1)^n \sin p\pi$.

$$\therefore a_n = \frac{\sin(p+n)\pi}{(p+n)\pi} + \frac{\sin(p-n)\pi}{(p-n)\pi} = \frac{(-1)^n}{\pi}\,\frac{2p\sin p\pi}{p^2-n^2}.$$

Now, $b_n = 0$ since $\cos px \sin nx$ is odd function.

Again $\cos px$ satisfies Dirichlet's Condition and since $\cos px$ is continuous everywhere therefore its Fourier Expansion is

$$\cos px = \frac{\sin p\pi}{p\pi} + \sum_{n=1}^{\infty}\left\{\frac{(-1)^n 2p\sin p\pi}{\pi(p^2-n^2)}\cos nx + 0\cdot \sin nx\right\}$$

$$\text{or, } \cos px = \frac{\sin p\pi}{p\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n 2p\sin p\pi}{\pi(p^2-n^2)}\cos nx$$

Putting $x=0$ we get

$$1 = \frac{\sin p\pi}{p\pi} + \sum_{n=1}^{\infty}\frac{(-1)^n 2p\sin p\pi}{\pi(p^2-n^2)} = \frac{\sin p\pi}{p\pi} + \frac{2p\sin p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$= \frac{\sin p\pi}{\pi}\left\{\frac{1}{p} - \frac{1}{p+1} - \frac{1}{p-1} + \frac{1}{p+2} + \frac{1}{p-2} - \frac{1}{p+3} - \frac{1}{p-3} + \ldots\ldots\right\}$$

$$\text{or, } \frac{\pi}{\sin p\pi} = \sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{n+p} + \frac{1}{n+1-p}\right).$$

Ex. 7. *Find a Fourier series to represent x^2 in the interval $(-l,l)$.*

The Fourier Co-efficients,

$$a_0 = \frac{1}{l}\int_{-l}^{l} x^2 dx = -\frac{2l^2}{3}$$

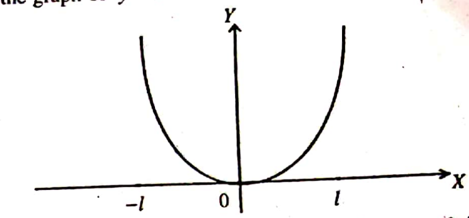
$$a_n = \frac{1}{l}\int_{-l}^{l} x^2 \cos nx\,dx = \frac{2}{l}\int_{0}^{l} x^2 \cos nx\,dx \quad (\because x^2\cos nx \text{ is even}) \text{ for } n \neq 0$$

$$= \frac{4l^2 \cos n\pi}{n^2\pi^2} \quad \text{[doing integ by part twice]}$$

$$= \frac{4l^2(-1)^n}{n^2\pi^2},\ n \neq 0.$$

and $b_n = \frac{1}{l}\int_{-l}^{l} x^2 \sin nx\,dx = 0$ $\quad \because x^2 \sin nx$ is odd function.

Again the graph of $y = x^2$ is shown in the following figure (it is a parabola).



This is bounded in $[-l,l]$ and monotonic in the two sub-interval $(-l,0)$ and $(0,l)$. So it satisfies Dirichlet's Condition. Since it is continuous everywhere so

$$x^2 = \frac{1}{2}\left(-\frac{2l^2}{3}\right) + \sum_{n=1}^{\infty}\left(\frac{4l^2(-1)^n}{n^2\pi^2}\cos nx + 0\cdot \sin nx\right)$$

or, $x^2 = -\frac{l^2}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n \cos nx}{n^2}$ which is the required Fourier Expansion.

Ex. 8. *Find the Fourier series of the function e^{-x} in the inerval $0 < x < 2\pi$.*

The interval is changed.

Here $c=0$, $c+2T = 2\pi$ $\therefore T = \pi$, $\therefore$ the Fourier co-efficients are

$$a_0 = \frac{1}{\pi}\int_{0}^{0+2\pi} e^{-x}dx = \frac{1-e^{-2\pi}}{\pi}$$

$$\text{and } a_n = \frac{1}{\pi}\int_0^{0+2\pi} e^{-x}\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[e^{-x}\frac{\sin nx}{n}\right]_0^{2\pi} + \int_0^{2\pi} e^{-x}\frac{\sin nx}{n}dx\right\}$$

$$= \frac{1}{\pi}\cdot\frac{1}{n}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{n\pi}\left\{-\left[e^{-x}\frac{\cos nx}{n}\right]_0^{2\pi} - \int_0^{2\pi} e^{-x}\frac{\cos nx}{n}dx\right\}$$

$$= \frac{1}{n\pi}\left\{-\left(e^{-2\pi}\frac{\cos 2n\pi}{n} - \frac{1}{n}\right) - \frac{1}{n}\int_0^{2\pi} e^{-x}\cos nx\,dx\right\}$$

TOPIC : POWER SERIES

SUBJECT: MATHEMATICS

SEMESTER: 4, CORE COURSE: 8

NAME OF TEACHER: PROF. AMIT SARKAR

Power Series.

A series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \quad (a_i \in \mathbb{R})$$

is called a ~~power~~ (real) power series. The numbers a_i $(i=0,1,2,3,\ldots)$ are known as the co-efficients. We will assume that if a series doesnot converge then it diverges.

We observe the following facts.

(i) Every power series converges for $x=0$

(ii) A power series may converge for some values of x and may diverge for some values of x. For example, the series $\sum_{n=0}^{\infty} x^n$ diverges if $|x| \geq 1$ and converges if $|x| < 1$.

(iii) A power series may converge for all x, e.g., the series $\sum_{n=0}^{\infty} \frac{x^n}{\underline{|n}}$ converges $\forall x \in \mathbb{R}$

(iv) A power series may diverge $\forall x (\neq 0)$ e.g. the series $\sum_{n=0}^{\infty} \underline{|n}\, . x^n$ diverges $\forall x (\neq 0) \in \mathbb{R}$.

Theorem If a power series converges for some $x = x_0$, then it converges absolutely $\forall x_1$ such that $|x_1| < |x_0|$.

Proof As $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_0$, $\sum_{n=0}^{\infty} a_n x_0^n$ is convergent and so $\lim_{n\to\infty} a_n x_0^n = 0$. This however means that the sequence $\{a_n x_0^n\}$ is is convergent. So, $\exists$ a constant $M\,(>0)$ such that

$$|a_n x_0^n| < M$$

i.e., $|a_n x_1^n| = |a_n (\frac{x_1}{x_0})^n x_0^n| = |a_n x_0^n|\,|\frac{x_1}{x_0}|^n$

$< M r^n$ where $r = |\frac{x_1}{x_0}| < 1$.

As $\sum M r^n$ is a geometric series with common ratio $r < 1$, it is convergent and so by comparison test $\sum_{n=0}^{\infty} |a_n x_1^n|$ is convergent i.e., $\sum_{n=0}^{\infty} a_n x_1^n$ is absolutely convergent.

<u>Theorem</u> If a power series diverges (does not converge) for some $x = x_0$ then it diverges $\forall x_1$ such that $|x_1| > |x_0|$.

<u>Proof</u> If the power series $\sum_{n=0}^{\infty} a_n x^n$ does not diverge (i.e., converges) for $x = x_1$ then by the preceding theorem it converges for $x = x_0$.

Suppose that a power series $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x = r$. This means that $\sum_{n=0}^{\infty} a_n r^n$ is convergent & so $\exists$ a constant $M\,(>0)$ such that

$$|a_n r^n| < M$$

i.e., $|a_n| < \frac{M}{r^n}$, i.e., $|a_n|^{1/n} < \frac{M^{1/n}}{r} < \frac{M}{r}$.

This however means that the sequence $\{|a_n|^{1/n}\}$ is bounded and so $\mu = \limsup_n |a_n|^{1/n}$ exists finitely. Obviously $\mu \geq 0$.

First we suppose that $\mu > 0$. Let $x\,(\neq 0) \in \mathbb{R}$ such that $|x| < \frac{1}{\mu}$ and $x_0 \in \mathbb{R}$ such that $|x| < x_0 < \frac{1}{\mu}$.

Then, $\mu = \limsup_n |a_n|^{1/n} < \frac{1}{x_0}$. So, $\exists$ a positive integer N such that

$|a_n|^{1/n} < \frac{1}{x_0}$ if $n \geq N$.

i.e. $|a_n|^{1/n} x_0 < 1$ if $n \geq N$.

i.e. $|a_n x_0^n| < 1$ if $n \geq N$.

So, if $n \geq N$,

$$|a_n x^n| = |a_n x_0^n (\frac{x}{x_0})^n| = |a_n x_0^n|\,|\frac{x}{x_0}|^n < \rho^n$$

where $\rho = |\frac{x}{x_0}| < 1$. Now, as $\sum \rho^n$ is convergent it follows that $\sum_{n=0}^{\infty} |a_n x^n|$ is convergent i.e., $\sum_{n=0}^{\infty} a_n x^n$ is absolutely convergent if $|x| < \frac{1}{\mu}$.

Suppose now that $x\,(\neq 0) \in \mathbb{R}$ is such that $|x| > \frac{1}{\mu}$.

Then, $\mu > \frac{1}{|x_0|}$ i.e.,

$\limsup_n |a_n|^{1/n} > \frac{1}{|x_0|}$

So, $|a_n|^{1/n} > \frac{1}{|x|}$ for an infinity of values of.

i.e., $|a_n x^n| > 1$ for an infinity of values of n.
So, the sequence $\{a_n x^n\}$ cannot converge to 0.
Hence if $|x| > \frac{1}{\mu}$, $\sum a_n x^n$ cannot converge.

Suppose now that $\mu = \limsup_n |a_n|^{1/n} = 0$.
Let $x (\neq 0) \in \mathbb{R}$. Then

$$\mu = \limsup_n |a_n|^{1/n} < \frac{1}{2|x|}.$$

and so $\exists$ a positive integer N such that

$$|a_n|^{1/n} < \frac{1}{2|x|} \quad \text{if } n \geq N$$

i.e., $$|a_n| < \frac{1}{2^n |x^n|} \quad \text{if } n \geq N$$

i.e., $$|a_n x^n| < \frac{1}{2^n} \quad \text{if } n \geq N$$

As $\sum\limits_{\infty} \frac{1}{2^n}$ is convergent, it follows that $\sum\limits_{n=0} a_n x^n$ converges absolutely. Thus if $\mu = 0$ then $\forall x (\neq 0) \in \mathbb{R}$ the series $\sum\limits_{n=0}^{\infty} a_n x^n$ converges absolutely.

Now, suppose that the sequence $\{|a_n|^{1/n}\}$ is unbounded above. Then, $\mu = \limsup_n |a_n|^{1/n} = \infty$. Now, in this case the series $\sum a_n x^n$ cannot converge for any $x (\neq 0) \in \mathbb{R}$.

We therefore obtain the following theorem :-

Theorem (Cauchy - Hadamard) Let $\sum a_n x^n$ be a given power series and $\mu = \limsup_n |a_n|^{1/n}$. If
(i) $\mu = 0$, then $\sum a_n x^n$ converges absolutely everywhere,
(ii) if $\mu = \infty$, then $\sum a_n x^n$ converges nowhere except $x = 0$, and if
(iii) $0 < \mu < \infty$, the series converges absolutely $\forall x \in \mathbb{R}$ such that $|x| < \frac{1}{\mu}$ and doesnot converge (i.e., diverges) for all those x such that $|x| > \frac{1}{\mu}$.

Observation Suppose $\sum a_n x^n$ is a power series and $r = \frac{1}{\mu} = \frac{1}{\limsup_n |a_n|^{1/n}}$. Then the series converges absolutely if $|x| < r$ and diverges if $|x| > r$. If $\mu = 0$ (i.e., $r = +\infty$) then the series converges absolutely $\forall x \in \mathbb{R}$ and the series doesnot converge for any $x (\neq 0) \in \mathbb{R}$ if $r = 0$ (i.e., $\mu = \infty$) i.e., if the sequence $\{|a_n|^{1/n}\}$ is unbounded above.

The number $r = \frac{1}{\mu} = \frac{1}{\limsup_n |a_n|^{1/n}}$ is known as the radius of convergence of the given power series and the open interval $(-r, r) = \{x : -r < x < r\}$ is known as the interval of convergence of the power series. Now, if $\lim\limits_{n \to \infty} \frac{a_{n+1}}{a_n} = l$, a finite number, then (by Cauchy's Second Theorem on Limits) $\lim_{n\to\infty} a_n^{1/n} = l$ and so, $\lim_{n\to\infty} |a_n|^{1/n} = |l|$. Thus, if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$, then $\frac{1}{|l|}$ is the radius of convergence of the power series $\sum a_n x^n$.

Properties of power series within the interval of convergence. Suppose $\sum_{n=0}^{\infty} a_n x^n$ is a power series having a positive radius of convergence say ρ. This means that $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $|x| < \rho$. So, if $x \in (-\rho, \rho)$ then $\sum_{n=0}^{\infty} a_n x^n$ defines a function of x, say, $f(x)$.

Let $\varepsilon (>0) \in \mathbb{R}$ such that $\varepsilon < \rho$. Then $\varepsilon < \rho - \varepsilon < \rho$ and clearly $\sum_{n=0}^{\infty} |a_n(\rho-\varepsilon)^n|$ is convergent. For each positive integer n, let $M_n = |a_n(\rho-\varepsilon)^n|$. Then $\sum M_n$ is a convergent series of positive terms. Let $\eta > 0$ be arbitrary. Then $\exists$ a positive integer N such that if $n \geq N$,

$$M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \eta$$

$\forall p = 1, 2, 3, \ldots$

Now, if $x \in [-\rho+\varepsilon, \rho-\varepsilon]$, then for all n

$$|a_n x^n| = |a_n|\,|x|^n \leq |a_n|\,|(\rho-\varepsilon)|^n = |a_n(\rho-\varepsilon)^n| = M_n$$

i.e., $|f_n(x)| \leq M_n$ $\forall n$ where $f_n(x) = a_n x^n$.

So, if $n \geq N$, then $\forall p = 1, 2, 3, \ldots$ & $\forall x \in [-\rho+\varepsilon, \rho-\varepsilon]$

$$|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p}|$$

$$= |f_{n+1}(x) + f_{n+2}(x) + \cdots + f_{n+p}(x)|.$$

$$\leq M_{n+1} + M_{n+2} + \cdots + M_{n+p} < \eta.$$

This however means that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly in $[-\rho+\varepsilon, \rho-\varepsilon]$.

As $\varepsilon > 0$ $(\varepsilon < \rho)$ is arbitrary, it follows $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly and absolutely in $(-\rho, \rho)$. We therefore obtain

Theorem A power series converges absolutely and uniformly within it's interval of convergence.

~~**Note.** If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f(x)$ is called the sum function of the power series. Now, as each $a_n x^n$ is continuous in $(-\rho, \rho)$ and $\sum a_n x^n$ converges uniformly to $f(x)$ in $(-\rho, \rho)$ it follows that $f(x)$ is continuous in $(-\rho, \rho)$.~~

Let ρ be the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$ and $-\rho < a < b < \rho$. Then we have seen that the power series converges absolutely and uniformly in $[a,b]$. Let $S_n(x) = \sum_{t=0}^{n} a_t x^t$; $x \in [a,b]$ and $f(x)$ denote sum function of $\sum_{n=0}^{\infty} a_n x^n$ in $[a,b]$. Now, as each $a_t x^t$ is continuous in $[a,b]$, it follows that the sequence $\{S_n(x)\}$ of continuous functions converges uniformly to $f(x)$ in $[a,b]$. Hence $f(x)$ must be continuous in $[a,b]$. We therefore obtain

Theorem A power series represents a continuous sum function within its interval of convergence.

Suppose that $\rho\ (>0)$ is the radius of convergence of a power series $\sum_{t=0}^{\infty} a_t x^t$ and $-\rho < a < b < \rho$. Then it is known that the given power series represents a continuous function $f(x)$ in $[a,b]$ & so $f(x)$ is integrable in $[a,b]$. In fact if $S_n(x) = \sum_{t=0}^{n} a_t x^t$, then the sequence $\{S_n(x)\}$ converges absolutely and uniformly to $f(x)$ in $[a,b]$. Now, $S_n(x)$ being the sum of continuous functions is continuous and hence integrable over the closed interval $[a,b]$.

Let $\varepsilon > 0$ be arbitrary. Then $\exists$ a positive integer N (independent of x in $[a,b]$) such that if $n \geq N$

$$|S_n(x) - f(x)| < \frac{\varepsilon}{b-a} \quad \forall x \in [a,b].$$

So, if $n \geq N$

$$\left| \int_a^b S_n(x)\,dx - \int_a^b f(x)\,dx \right| = \left| \int_a^b \{S_n(x) - f(x)\}\,dx \right|$$

$$\leq \int_a^b |S_n(x) - f(x)|\,dx < (b-a)\cdot\frac{\varepsilon}{b-a} = \varepsilon$$

This however means that

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b S_n(x)\,dx$$

i.e., $$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b \left(\sum_{t=0}^{n} a_t x^t \right) dx$$

$$= \lim_{n\to\infty} \sum_{t=0}^{n} \int_a^b a_t x^t\,dx = \sum_{t=0}^{\infty} \int_a^b a_t x^t\,dx$$

We therefore obtain the following theorem:

Theorem A power series may be integrated term by term in any closed interval that lies entirely within the interval of convergence.

That is, suppose $\rho\,(>0)$ is the radius of convergence of the power series $\sum_{t=0}^{\infty} a_t x^t$ and let $f(x) = \sum_{t=0}^{\infty} a_t x^t$, $|x| < \rho$. If $-\rho < a < b < \rho$ then

$$\int_a^b f(x)\,dx = \int_a^b a_0\,dx + \int_a^b a_1 x\,dx + \cdots + \int_a^b a_n x^n\,dx + \cdots$$

Ex:- Suppose $\{a_n\}$ and $\{b_n\}$ are bounded sequences of positive numbers. Then

(i) $\overline{\lim}\,(a_n b_n) \geq (\underline{\lim}\, a_n)(\overline{\lim}\, b_n)$

and (ii) $\underline{\lim}\,(a_n b_n) \leq (\underline{\lim}\, a_n)(\overline{\lim}\, b_n)$.

Solⁿ: As $\{a_n\}$ and $\{b_n\}$ are bounded sequences of positive numbers, $\exists$ constants $m\,(>0)$ and $M>0$ such that $\forall n$, $m \leq a_n \leq M$ and $m \leq b_n \leq M$. Let $a = \underline{\lim}\, a_n$, $A = \overline{\lim}\, a_n$, $b = \underline{\lim}\, b_n$, $B = \overline{\lim}\, b_n$ and $\varepsilon > 0$ be arbitrary.

(i) Then $\exists$ a positive integer N such that

$$a_n > a - \frac{\varepsilon}{2(B+1)} \text{ if } n \geq N$$

& for an infinity of values of n,

$$b_n > b - \frac{\varepsilon}{2(m+1)}.$$

So, for an infinity of values of n,

$$a_n b_n - aB = a_n b_n - a_n B + a_n B - aB = a_n(b_n - B) + B(a_n - a)$$
$$\geq m(b_n - B) + B(a_n - a) > m\left\{-\frac{\varepsilon}{2(m+1)}\right\} + B\left\{-\frac{\varepsilon}{2(B+1)}\right\}$$
$$> -\varepsilon$$

Thus, for an infinity of values of n

$$a_n b_n > aB - \varepsilon$$

As $\varepsilon > 0$ is arbitrary we see that

$a_n b_n \geq aB$ for an infinity of values of n

Hence

$$\overline{\lim}\,(a_n b_n) \geq aB = (\underline{\lim}\, a_n)(\overline{\lim}\, b_n).$$

(ii) Then $\exists$ a positive integer N such that $n \geq N \Rightarrow$

$$b_n < B + \frac{\varepsilon}{2M}$$

and for an infinity of values of n

$$a_n < a_n + \frac{\varepsilon}{2(B+1)}$$

Hence, for an infinity of values of n

$$a_n b_n - aB = a_n(b_n - B) + B(a_n - a) < M\cdot\frac{\varepsilon}{2M} + B\cdot\frac{\varepsilon}{2(B+1)} < \varepsilon$$

i.e., $a_n b_n < aB + \varepsilon$, for an infinity of values of n.

As $\varepsilon > 0$ is arbitrary,

$a_n b_n \leq aB$ for an infinity of values of n.

So,

$$\underline{\lim}(a_n b_n) \leq aB = (\underline{\lim}\, a_n)(\overline{\lim}\, b_n)$$

Note By a known result and the above example.

$$aB \leq \overline{\lim}(a_n b_n) \leq AB$$

Now if $\{a_n\}$ be convergent then $a = A$ and so

$$\overline{\lim}(a_n b_n) = (\lim a_n)(\overline{\lim}\, b_n)$$

provided $\{a_n\}$ is convergent.

Suppose now that ρ is the radius of convergence of a power series

$$\sum_{t=0}^{\infty} a_t x^t = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots \quad \text{---}(1)$$

Let $f(x) = \sum_{t=0}^{\infty} a_t x^t \;;\; |x| < \rho$

In fact if $S_n(x) = \sum^{n} a_t x^t$, then the sequence $\{S_n(x)\}$ converges uniformly to $f(x)$ in the closed interval $[a, b]$ where $-\rho < a < b < \rho$. We consider the power series

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots + n a_n x^{n-1} + \cdots = \sum_{t=1}^{\infty} t a_t x^{t-1} \quad \text{---}(2)$$

obtained by differentiating (w.r.t. x) the terms of the power series (1). We note that (2) is also a power series. Now, as $\lim_{n\to\infty} n^{1/n} = 1$,

$$\overline{\lim_{n\to\infty}} |n a_n|^{1/n} = \lim_{n\to\infty} n^{1/n} \cdot \overline{\lim} |a_n|^{1/n} = \overline{\lim_{n\to\infty}} |a_n|^{1/n}$$

So, if ρ' denotes the radius of convergence of the power series (2) then

$$\rho' = \frac{1}{\overline{\lim}\, |n a_n|^{1/n}} = \frac{1}{\overline{\lim_{n\to\infty}} |a_n|^{1/n}} = \rho$$

In other words the power series (1) and the power series (2) {that is the series obtained by differentiating each term of (1) or the differentiated power series} have the same radius of convergence.

Let $\phi(y) = \sum_{t=1}^{\infty} t a_t y^{t-1}$, $y \in [a, b]$ ---(3)

where $-\rho < a < b < \rho$. Now if $x \in (a, b]$ then* (by the preceding theorem)*

$$\int_a^x \phi(y)\,dy = \sum_{n=1}^{\infty} \int_a^x n a_n y^{n-1}\,dy = \sum_{n=1}^{\infty} (a_n x^n - a_n a^n)$$

i.e. $\int_a^x \phi(y)\,dy = f(x) - f(a)$

Now, as ϕ is continuous in $[a, b]$

$$\phi(x) = \frac{d}{dx}\int_a^x \phi(y)\,dy = f'(x)$$

* Since a power series may be integrated term by term in any closed interval that lies entirely within the interval of convergence, we have

i.e., $f'(x) = \sum_{t=1}^{\infty} t a_t x^{t-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots\cdots$

We therefore obtain the following theorem :-

Theorem (Differentiation of power series) A power series may be differentiated term by term in any closed interval which lies entirely within its interval of convergence. That is suppose $\rho\,(>0)$ is the radius of convergence of the power series $\sum_{t=0}^{\infty} a_t x^t$. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $|x| < \rho$ and $-\rho < a < b < \rho$ then f is differentiable on $[a, b]$ &

$$f'(x) = \sum_{t=1}^{\infty} t a_t x^{t-1}; \quad x \in [a, b].$$

Further both the power series have the same radii of convergence.

By repeated application of the preceding theorem, we conclude that if $k \in N$ (N being the set of all natural numbers), then $\sum_{n=0}^{\infty} a_n x^n$ can be differentiated term-by-term k times to obtain the power series

$$\sum_{n=k}^{\infty} \frac{\lfloor n}{\lfloor n-k} a_n x^{n-k}$$

~~Theorem~~ The radius of convergence of this power series is ρ and the above series converges uniformly & absolutely in any closed interval $[a, b]$ such that $-\rho < a < b < \rho$ to $f^{(k)}(x)$. That is

$$f^{(k)}(x) = \sum_{n=k}^{\infty} \frac{\lfloor n}{\lfloor n-k} a_n x^{n-k}; \quad x \in [a, b].$$

If we substitute $x = 0$ in the above we obtain

$$f^{(k)}(0) = (\lfloor k) \cdot a_k$$

Suppose now that $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ converge on the same interval $(-\rho, \rho) = \{x : -\rho < x < \rho\}$; $\rho > 0$ to the same function f. Then by our preceding remark.

$$(\lfloor n) a_n = f^n(0) = (\lfloor n) b_n \quad \forall n \in N.$$

& so $\quad a_n = b_n \quad \forall n \in N$

We therefore obtain the following theorem.

Theorem (Uniqueness Theorem) If $\sum_{n=0}^{\infty} a_n x^n$ & $\sum_{n=0}^{\infty} b_n x^n$ converge on some interval $-\rho < x < \rho$; $\rho > 0$ to the same function f then

$$a_n = b_n \quad \forall n \in N \text{ (N being the set of all natural nos)}$$

Suppose that ρ is the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$. Now if $-\rho < a < b < \rho$ then the above power series converges absolutely and uniformly in $[a, b]$ and determines a continuous function $f(x)$ in $[a, b]$. That is

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad ; \quad a \le x \le b$$

Suppose now that the given power series converges at $x = \rho$. We show that $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly in $[a, \rho]$. For all $p = 1, 2, 3, \dots$, Let

$$S_{n,p} = a_{n+1}\rho^{n+1} + a_{n+2}\rho^{n+2} + \dots + a_{n+p}\rho^{n+p}.$$

Then obviously,

$$a_{n+1}\rho^{n+1} = S_{n,1}, \quad a_{n+2}\rho^{n+2} = S_{n,2} - S_{n,1}, \quad a_{n+3}\rho^{n+3} = S_{n,3} - S_{n,2}, \quad \dots, \quad a_{n+p}\rho^{n+p} = S_{n,p} - S_{n,p-1}.$$

Let $\varepsilon > 0$ be arbitrary. Then $\exists$ a positive integer N such that $n \ge N$ implies

$$|a_{n+1}\rho^{n+1} + a_{n+2}\rho^{n+2} + \dots + a_{n+p}\rho^{n+p}| < \varepsilon$$

for all $p = 1, 2, 3, 4, \dots$

i.e. $|S_{n,p}| < \varepsilon$ if $n \ge N$ & $\forall\, p = 1, 2, 3, \dots$

Now if $x \in [0, \rho]$ then $0 \le x \le \rho$

$\left|\frac{x}{\rho}\right|$ so $0 \le \frac{x}{\rho} \le 1$.

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$$\left|\frac{x}{\rho}\right|^{n+p} \le \left|\frac{x}{\rho}\right|^{n+p-1} \le \left|\frac{x}{\rho}\right|^{n+p-2} \le \dots \le \left|\frac{x}{\rho}\right|^{n+1} \le 1.$$

Hence $\left(\frac{x}{\rho}\right)^2 = \frac{x}{\rho}\cdot\frac{x}{\rho} \le \frac{x}{\rho}$, $\left(\frac{x}{\rho}\right)^3 \le \left(\frac{x}{\rho}\right)^2$. Continuing in this way we see that $\left\{\left(\frac{x}{\rho}\right)^n\right\}$ is a monotone decreasing sequence. Now, if $x \in [0, \rho]$, then

$$a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + a_{n+3}x^{n+3} + \dots + a_{n+p-2}x^{n+p-2} + a_{n+p-1}x^{n+p-1} + a_{n+p}x^{n+p}$$

$$= a_{n+1}\rho^{n+1}\left(\frac{x}{\rho}\right)^{n+1} + a_{n+2}\rho^{n+2}\left(\frac{x}{\rho}\right)^{n+2} + a_{n+3}\rho^{n+3}\left(\frac{x}{\rho}\right)^{n+3} + \dots + a_{n+p-2}\rho^{n+p-2}\left(\frac{x}{\rho}\right)^{n+p-2} + a_{n+p-1}\rho^{n+p-1}\left(\frac{x}{\rho}\right)^{n+p-1} + a_{n+p}\rho^{n+p}\left(\frac{x}{\rho}\right)^{n+p}$$

$$\left.\begin{aligned}
&= \left(\frac{x}{\rho}\right)^{n+1}\{S_{n,1}\} \\
&+ \left(\frac{x}{\rho}\right)^{n+2}\{S_{n,2} - S_{n,1}\} \\
&+ \left(\frac{x}{\rho}\right)^{n+3}\{S_{n,3} - S_{n,2}\} \\
&+ \dots \\
&+ \left(\frac{x}{\rho}\right)^{n+p-2}\{S_{n,p-2} - S_{n,p-3}\} \\
&+ \left(\frac{x}{\rho}\right)^{n+p-1}\{S_{n,p-1} - S_{n,p-2}\} \\
&+ \left(\frac{x}{\rho}\right)^{n+p}\{S_{n,p} - S_{n,p-1}\}
\end{aligned}\right\}
= \left\{\begin{aligned}
&S_{n,1}\left\{\left(\frac{x}{\rho}\right)^{n+1} - \left(\frac{x}{\rho}\right)^{n+2}\right\} \\
&+ S_{n,2}\left\{\left(\frac{x}{\rho}\right)^{n+2} - \left(\frac{x}{\rho}\right)^{n+3}\right\} \\
&+ S_{n,3}\left\{\left(\frac{x}{\rho}\right)^{n+3} - \left(\frac{x}{\rho}\right)^{n+4}\right\} \\
&+ \dots \\
&+ S_{n,p-2}\left\{\left(\frac{x}{\rho}\right)^{n+p-2} - \left(\frac{x}{\rho}\right)^{n+p-1}\right\} \\
&+ S_{n,p-1}\left\{\left(\frac{x}{\rho}\right)^{n+p-1} - \left(\frac{x}{\rho}\right)^{n+p}\right\} \\
&+ S_{n,p}\left(\frac{x}{\rho}\right)^{n+p}
\end{aligned}\right.$$

Hence if $n \geq N$, then $\forall p = 1, 2, 3, \ldots$ & $x \in [0, \rho]$.

$|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + a_{n+3}x^{n+3} + \cdots + a_{n+p}x^{n+p}|$

$$= \left| S_{n,1}\left\{\left(\frac{x}{\rho}\right)^{n+1} - \left(\frac{x}{\rho}\right)^{n+2}\right\} + S_{n,2}\left\{\left(\frac{x}{\rho}\right)^{n+2} - \left(\frac{x}{\rho}\right)^{n+3}\right\} + S_{n,3}\left\{\left(\frac{x}{\rho}\right)^{n+3} - \left(\frac{x}{\rho}\right)^{n+4}\right\} + \cdots + S_{n,p-2}\left\{\left(\frac{x}{\rho}\right)^{n+p-2} - \left(\frac{x}{\rho}\right)^{n+p-1}\right\} + S_{n,p-1}\left\{\left(\frac{x}{\rho}\right)^{n+p-1} - \left(\frac{x}{\rho}\right)^{n+p}\right\} + S_{n,p}\left(\frac{x}{\rho}\right)^{n+p} \right|$$

$$< \varepsilon \left[\left(\frac{x}{\rho}\right)^{n+1} - \left(\frac{x}{\rho}\right)^{n+2} + \left(\frac{x}{\rho}\right)^{n+2} - \left(\frac{x}{\rho}\right)^{n+3} + \left(\frac{x}{\rho}\right)^{n+3} - \left(\frac{x}{\rho}\right)^{n+4} + \cdots + \left(\frac{x}{\rho}\right)^{n+p-2} - \left(\frac{x}{\rho}\right)^{n+p-1} + \left(\frac{x}{\rho}\right)^{n+p-1} - \left(\frac{x}{\rho}\right)^{n+p} + \left(\frac{x}{\rho}\right)^{n+p} \right] = \varepsilon \left(\frac{x}{\rho}\right)^{n+p} \leq \varepsilon$$

i.e., $|a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots + a_{n+p}x^{n+p}| < \varepsilon$ if $n \geq N$ & $\forall p = 1, 2, 3, \ldots$ & $\forall x \in [0, \rho]$.

Hence the given power series converges uniformly in $[0, \rho]$.

We therefore obtain the following theorem.

Theorem. If a power series $\sum_{n=0}^{\infty} a_n x^n$ with $\rho (>0)$ as its radius of convergence converges at the point ρ then the power series converges uniformly in the interval $[0, \rho]$.

Now by the above theorem it follows that if a power series with radius of convergence $\rho (>0)$ converges at $x = \rho$ then it converges uniformly to a continuous function $F(x)$ in the closed interval $[-\rho+\delta, \rho]$ where $\delta > 0$ is arbitrarily small.

A similar situation arises when the series converges at $x = -\rho$.

Ex. Find the radius of convergence of the following series:—

(i) $x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$

Soln. Comparing with $\sum a_n x^n$ we see that $a_n = \frac{1}{\lfloor n}$

So, $\frac{a_{n+1}}{a_n} = \frac{\lfloor n}{\lfloor n+1} = \frac{1}{n+1} \to 0$ as $n \to \infty$

Hence $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \overline{\lim_{n\to\infty}} a_n^{1/n} = \lim_{n\to\infty} a_n^{1/n} = 0$.

So, R = radius of convergence of the given series is $+\infty$. In otherwords the series converges everywhere.

(ii) $\frac{1}{2}x + \frac{1\cdot3}{2\cdot5}x^2 + \frac{1\cdot3\cdot5}{2\cdot5\cdot8}x^3 + \cdots$ (70)

Comparing with $\sum a_n x^n$ we see that $a_n = \frac{1.3.5\cdots(2n-1)}{2.5.8\cdots(3n-1)}$

So, $a_{n+1} = \frac{1.3.5\cdots(2n-1)(2n+1)}{2.5.8\cdots(3n-1)(3n+2)} = a_n \cdot \frac{2n+1}{3n+2}$

Hence $\frac{a_{n+1}}{a_n} = \frac{2n+1}{3n+2} \to \frac{2}{3}$ as $n \to \infty$.

Hence the radius of convergence of the given series is $3/2$. ~~and~~

(iii) $x + \frac{x^2}{2^2} + \frac{\lfloor 2}{3^2}x^3 + \frac{\lfloor 3}{4^4}x^4 + \cdots$

Comparing with the series $\sum a_n x^n$ we see that

$a_n = \frac{\lfloor n-1}{n^n}$ & so $a_{n+1} = \frac{\lfloor n}{(n+1)^{n+1}}$. So,

$$\frac{a_{n+1}}{a_n} = \frac{\lfloor n}{(n+1)^{n+1}} \cdot \frac{n^n}{\lfloor n-1} = \frac{n^{n+1}}{(n+1)^{n+1}} = \frac{1}{\left(\frac{n+1}{n}\right)^{n+1}}$$

$$= \frac{1}{\left(1+\frac{1}{n}\right)^n} \cdot \frac{1}{1+\frac{1}{n}} \to \frac{1}{e} \text{ as } n \to \infty$$

So, the radius of convergence of the given series is e

(iv) $\sum \frac{x^n}{n^2}$.

Here, $a_n = \frac{1}{n^2}$, $a_{n+1} = \frac{1}{(n+1)^2}$

So, $\frac{a_{n+1}}{a_n} = \frac{n^2}{(n+1)^2} \to 1$ as $n \to \infty$.

Hence the radius of convergence of the given series is 1.

Fourier Series

Definition   A function $f(x)$ is called periodic if $\exists$ a constant $T>0$ such that

$$f(x+T) = f(x)$$

for any x in the domain of definition of x.

If T is a period of the function $f(x)$, then $2T, 3T, 4T, \ldots$ are also periods because

$f(x) = f(x+T) = f(x+2T) = f(x+3T) = \cdots$.

Also if T is a period of $f(x)$, then the relation

$f(x) = f(x-T+T) = f(x-T)$ {as T is a period}

shows that $-T$ is also a period. Further,

$f(x-T) = f(x-2T+T) = f(x-2T) = \cdots$.

Thus if T is a period of $f(x)$ then for any integer k, kT is also a period.

From the above discussions we see that the period of a function is not unique.

Some basic properties

Let f be a periodic function with period $2T$ $(T>0)$. Let $f \in R[-T,T]$ i.e., f be Riemann integrable in $[-T,T]$. We note that if ~~$a \in \mathbb{R}$ then~~ $a>0$, then