Infinite series (G.S.D)

Sem-(II), CC-3, Unit(3)

D'Alembert's Ratio test

(7) Test the series $1+\frac{x}{2}+\frac{x^2}{5}+\frac{x^3}{10}+\cdots$

Sol: Here $u_n = \frac{x^n}{n^2+1}$ (omitting the 1st term)

$\therefore u_{n+1} = \frac{x^{n+1}}{(n+1)^2+1} = \frac{x^{n+1}}{n^2+2n+2}$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{n^2+2n+2} \times \frac{n^2+1}{x^n}$

$= \lim_{n\to\infty} \frac{n^2+1}{n^2+2n+2}\, x$

$= \lim_{n\to\infty} \frac{1+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} \cdot x = \frac{1+0}{1+0+0} \cdot x = x$

Hence by D'Alembert's Ratio Test, the given series is convergent when $x<1$ and divergent when $x>1$. But this test fails when $x=1$

When $x=1$, $u_n = \frac{1}{n^2+1} = \frac{1}{n^2\left(1+\frac{1}{n^2}\right)}$

Let $v_n = \frac{1}{n^2}$

Now, $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1$, which is finite

Now $\Sigma v_n = \Sigma \frac{1}{n^2}$ is a p-series $\Sigma \frac{1}{n^p}$ where $p=2>1$

So $\Sigma v_n = \Sigma \frac{1}{n^2}$ is convergent

So by Comparison Test Σu_n is convergent when $x=1$

Hence the given series is convergent when $x \leq 1$. and div when $x>1$.

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(8) Test the series $1 + \frac{2^p}{\lfloor 2} + \frac{3^p}{\lfloor 3} + \frac{4^p}{\lfloor 4} + \cdots \quad (p>0)$. (WB 20)

Sol Here $u_n = \frac{n^p}{\lfloor n}$ $\therefore u_{n+1} = \frac{(n+1)^p}{\lfloor n+1}$

So, $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{\lfloor n+1} \times \frac{\lfloor n}{n^p}$

$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^p \cdot \frac{1}{n+1}$

$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p \cdot \frac{\frac{1}{n}}{1+\frac{1}{n}}$

$= (1+0)^p \cdot \frac{0}{1+0} = 0 < 1$

Hence by D'Alembert's Ratio test the given series is Convergent ✓

(9) Test the Convergence of the series $\sum_{n=1}^{\infty} \frac{\lfloor n \cdot 2^n}{n^n}$.

Sol Here $u_n = \frac{\lfloor n\, 2^n}{n^n}$

$u_{n+1} = \frac{\lfloor n+1\, 2^{n+1}}{(n+1)^{n+1}}$

$\therefore \frac{u_{n+1}}{u_n} = \frac{\lfloor n+1 \cdot 2^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{\lfloor n\, 2^n} = \frac{(n+1)\, 2 \cdot n^n}{(n+1)(n+1)^n}$

$= \frac{2}{\left(\frac{n+1}{n}\right)^n} = \frac{2}{\left(1+\frac{1}{n}\right)^n}$

$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1$

$(\because 2 < e < 3$
$\therefore \frac{2}{e} < 1)$

So, by D'Alembert's Ratio test ✓ the given series is Convergent ✓

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(10) Test the series

$$1+\frac{2^2}{3^2}x+\frac{2^2.4^2}{3^2.5^2}x^2+\frac{2^2.4^2.6^2}{3^2.5^2.7^2}x^3+\cdots \quad (x\neq 1)$$

Sol Here $u_n = \frac{2^2.4^2.6^2\cdots(2n)^2}{3^2.5^2.7^2\cdots(2n+1)^2}x^n$ [omitting the first term]

$$\therefore u_{n+1} = \frac{2^2.4^2.6^2\cdots(2n+2)^2}{3^2.5^2.7^2\cdots(2n+3)^2}x^{n+1}$$

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{2^2.4^2.6^2\cdots(2n+2)^2}{3^2.5^2.7^2\cdots(2n+3)^2}x^{n+1}\times\frac{3^2.5^2\cdots(2n+1)^2}{2^2.4^2\cdots(2n)^2}\cdot\frac{1}{x^n}$$

$$=\lim_{n\to\infty}\frac{(2n+2)^2}{(2n+3)^2}x = \lim_{n\to\infty}\frac{\left(2+\frac{2}{n}\right)^2}{\left(2+\frac{3}{n}\right)^2}\cdot x$$

$$=\frac{(2+0)^2}{(2+0)^2}\cdot x = x$$

Hence ~~the given series~~ by D'Alembert's Ratio test, the given series is convergent when $x<1$ and divergent when $x>1$

(11) Discuss the convergency of the series

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+\cdots$$

Sol Here $u_n = \frac{x^{2n-2}}{(n+1)\sqrt{n}}$, $\therefore u_{n+1} = \frac{x^{2n}}{(n+2)\sqrt{n+1}}$

$$\therefore \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty}\frac{x^{2n}}{(n+2)\sqrt{n+1}}\times\frac{(n+1)\sqrt{n}}{x^{2n-2}}$$

$$=\lim_{n\to\infty}\frac{n+1}{n+2}\sqrt{\frac{n}{n+1}}\,x^2$$

$$=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\sqrt{\frac{1}{1+\frac{1}{n}}}\,x^2 = \frac{1+0}{1+0}\sqrt{\frac{1}{1+0}}\,x^2 = x^2$$

Hence by D'Alembert's Ratio test the given series is convergent when $x^2<1$ & divergent when $x^2>1$.

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When $x^2 = 1$ the test fails.

When $x^2 = 1$, $u_n = \frac{1}{(n+1)\sqrt{n}} = \frac{1}{n(1+\frac{1}{n})\sqrt{n}}$

$$= \frac{1}{n^{3/2}(1+\frac{1}{n})}$$

Let $v_n = \frac{1}{n^{3/2}}$

Now $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1$, which is finite

Also, $\sum v_n = \sum \frac{1}{n^{3/2}}$ is a p-series $\sum \frac{1}{n^p}$

where $p = 3/2 > 1$

So, $\sum v_n = \sum \frac{1}{n^{3/2}}$ is Convergent

So by comparison test, $\sum u_n$ is Convergent when $x^2 = 1$

Hence the given series is Convergent when $x^2 \le 1$ & divergent when $x^2 > 1$

(12) Test the convergence of the (+)ve term series

$$1 + \frac{\alpha+1}{\beta+1} + \frac{(\alpha+1)(2\alpha+1)}{(\beta+1)(2\beta+1)} + \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)} + \cdots$$

Sol Here $u_n = \frac{(\alpha+1)(2\alpha+1)(3\alpha+1)\cdots(n\alpha+1)}{(\beta+1)(2\beta+1)(3\beta+1)\cdots(n\beta+1)}$ (omitting the 1st term)

& $u_{n+1} = \frac{(\alpha+1)(2\alpha+1)\cdots(n\alpha+1)\{(n+1)\alpha+1\}}{(\beta+1)(2\beta+1)\cdots(n\beta+1)\{(n+1)\beta+1\}}$

$$\therefore \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left\{\frac{(\alpha+1)(2\alpha+1)\cdots(n\alpha+1)\{(n+1)\alpha+1\}}{(\beta+1)(2\beta+1)\cdots(n\beta+1)\{(n+1)\beta+1\}} \times \frac{(\beta+1)(2\beta+1)\cdots(n\beta+1)}{(\alpha+1)(2\alpha+1)\cdots(n\alpha+1)}\right\}$$

$$= \lim_{n\to\infty} \frac{(n+1)\alpha+1}{(n+1)\beta+1} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n}\right)\alpha+\frac{1}{n}}{\left(1+\frac{1}{n}\right)\beta+\frac{1}{n}}$$

$$= \frac{(1+0)\alpha+0}{(1+0)\beta+0} = \frac{\alpha}{\beta}$$

Hence by D'Alembert's ratio Test, the given series is convergent when $\frac{\alpha}{\beta}<1$ i.e. when $\alpha<\beta$ & divergent when $\frac{\alpha}{\beta}>1$ i.e. when $\alpha>\beta$. When $\frac{\alpha}{\beta}=1$ i.e. when $\alpha=\beta$ the test fails.

When $\alpha=\beta$ then $u_n=1$

then $\lim_{n\to\infty} u_n = 1 \neq 0$

So, by Cauchy's Fundamental test the given series is divergent when $\alpha=\beta$. ($\because \lim_{n\to\infty} u_n \neq 0$)

Hence the given series is convergent when $\alpha<\beta$ & divergent when $\alpha \geq \beta$.

13) Test the convergence of the series

$$\left(\frac{1}{3}\right)^2+\left(\frac{1\cdot2}{3\cdot5}\right)^2+\left(\frac{1\cdot2\cdot3}{3\cdot5\cdot7}\right)^2+\cdots$$

Sol Here $u_n = \left\{\frac{1\cdot2\cdot3\cdots n}{3\cdot5\cdot7\cdots(2n+1)}\right\}^2$

$\therefore u_{n+1} = \left\{\frac{1\cdot2\cdot3\cdots n(n+1)}{3\cdot5\cdot7\cdots(2n+1)(2n+3)}\right\}^2$

$$\therefore \lim_{n\to\infty}\left(\frac{u_{n+1}}{u_n}\right) = \lim_{n\to\infty}\left\{\frac{1\cdot2\cdot3\cdots n(n+1)}{3\cdot5\cdot7\cdots(2n+1)(2n+3)}\right\}^2 \times \left\{\frac{3\cdot5\cdot7\cdots(2n+1)}{1\cdot2\cdot3\cdots n}\right\}^2$$

$$= \lim_{n\to\infty}\frac{(n+1)^2}{(2n+3)^2} = \lim_{n\to\infty}\left(\frac{1+\frac{1}{n}}{2+\frac{3}{n}}\right)^2 = \frac{1}{2^2} = \frac{1}{4}<1$$

Hence by D'Alembert's Ratio Test, the given series is convergent

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Cauchy's Root Test

(14) Test the convergence of the series $\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$

Sol. Here $u_n = \left(\frac{n}{n+1}\right)^{n^2}$

$$\therefore \sqrt[n]{u_n} = \sqrt[n]{\left(\frac{n}{n+1}\right)^{n^2}} = \left(\frac{n}{n+1}\right)^n$$

$$\therefore \lim_{n\to\infty} \sqrt[n]{u_n} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}$$

$$= \frac{1}{e} < 1 \quad \left(\because 2<e<3 \quad \therefore e>1 \quad \therefore \tfrac{1}{e}<1\right)$$

So, by Cauchy's ~~Root~~ root test, the given series is convergent.

(15) Examine the nature of the series

$$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$$

Here $u_n = \left\{\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right\}^{-n}$

$$\therefore u_n^{1/n} = \left\{\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right\}^{-1}$$

$$= \left(\frac{n+1}{n}\right)^{-1}\left\{\left(\frac{n+1}{n}\right)^n - 1\right\}^{-1}$$

$$\therefore \lim_{n\to\infty} u_n^{1/n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^{-1}\left\{\left(\frac{n+1}{n}\right)^n - 1\right\}^{-1}$$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{-1}\left\{\left(1+\frac{1}{n}\right)^n - 1\right\}^{-1}$$

$$= (1+0)^{-1}\cdot(e-1)^{-1} \quad \left[\because \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e\right]$$

$$= \frac{1}{e-1} < 1 \quad \left[\because 2<e<3 \quad \therefore 1<e-1<2 \quad \therefore \frac{1}{e-1}<1\right]$$

So by Cauchy's root test the given series is Convergent.