$\ldots dx > 1 + \sqrt{1-c^2}$.

5.9. PRIMITIVES AND FUNDAMENTAL THEOREMS.

Definition (Primitive) : Let $f : [a, b] \to \mathbb{R}$ be a given function. A function $F : [a, b] \to \mathbb{R}$ is said to be primitive or antiderivative of f on $[a, b]$ if F is continuous on $[a, b]$, differentiable on (a, b) and $\frac{d}{dx}F(x) = f(x)$, $x \in (a, b)$.

The function F defined by $F(x) = \int_a^x f(t)dt$ $(x \in [a, b])$ is called the indefinite integral of f on $[a, b]$.

For example, $\log_e x$ $(x \in [1, 2])$ is the primitive of $\frac{1}{x}$ on $[1, 2]$. The integral $\int_a^x \frac{dt}{t}$ $(x \in [1, 2])$ is the indefinite integral of $\frac{1}{t}$ on $[1, 2]$.

Remark : One may be tempted to jump into the conclusions :

(i) if f is integrable on $[a, b]$ then it has a primitive on $[a, b]$,

(ii) if f has *a* primitive on $[a, b]$ then f is integrable on $[a, b]$.

The following two examples show that such conclusions are false.

***Example* 5.9.1.** Let $f : [0, 2] \to \mathbb{R}$ be a function defined by

$$f(x) = 2x, \quad 0 \le x \le 1$$
$$= x^2, \quad 1 < x \le 2$$

Show that f has no primitive although f is integrable on $[0, 2]$.

***Solution* :** The function f is continuous on $[0, 2]$ except at the point $x = 1$. Hence f is integrable on $[0, 2]$. f has a jump discontinuity at $x = 1$, since $f(1 + 0) \neq f(1 - 0)$. As we know that derivative of a function (assuming its existence) can not have points of jump discontinuity in its domain, it follows that there does not exist a function F on $[0, 2]$ such that $F'(x) = f(x)$, $x \in (0, 2)$. Hence f has no primitive on $[0, 2]$ although it is integrable there.

Example 5.9.2. Let $f : [-1, 1] \to \mathbb{R}$ be a function defined by

$$f(x) = 2x \sin\frac{1}{x^3} - \frac{3}{x^2}\cos\frac{1}{x^3}, \quad x \neq 0$$

and $$f(0) = 0.$$

Show that f has primitive in $[-1, 1]$ but $f \notin \mathcal{R}[-1, 1]$.

Solution : Since $\lim_{x\to 0} f(x)$ is infinite, 0 is a point of infinite discontinuity of f in $[-1, 1]$ and hence f is unbounded on $[-1, 1]$.

Thus f is not integrable on $[-1, 1]$.

If we define a function F on $[-1, 1]$ by

$$F(x) = x^2 \sin\frac{1}{x^3}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

then $$F'(x) = 2x \sin\frac{1}{x^3} - \frac{3}{x^2}\cos\frac{1}{x^3}, \quad x \neq 0$$
$$= 0, \quad x = 0$$

[$F'(0) = \lim_{h\to 0}\frac{F(h) - F(0)}{h} = \lim_{h\to 0} h\sin\frac{1}{h^3} = 0$ as $\sin\frac{1}{h^3}$ is bounded in any deleted neigbourhood of origin].

Thus F is differentiable everywhere on $[-1, 1]$ and $F'(x) = f(x)$, $x \in [-1, 1]$. Therefore F is a primitive of f on $[-1, 1]$ although f is not integrable there.

Note : If F be a primitive of f on $[a, b]$ then for any $c \in \mathbb{R}$, $F + c$ is also a primitive of f on $[a, b]$. Thus primitive of f, if exists, is not unique.

Theorem 5.9.1.

Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, $x \in [a, b]$. Then

(i) F is continuous on $[a, b]$,

(ii) F is of bounded variation on $[a, b]$.

Proof. (i) Let c be any point of $[a, b]$. Then for any $x \in [a, b]$ $(x \geq c$ or $x < c)$

$$F(x) - F(c) = \int_a^x f(t)dt - \int_a^c f(t)dt = \int_c^x f(t)dt \quad \cdots (1)$$

Since f is R-integrable on $[a, b]$, it is bounded on $[a, b]$. Hence there exists a positive real number A such that

$$|f(t)| \leq A \text{ for } t \in [a, b] \quad ... (2)$$

Then for $x \geq c$, we have from (1)

$$|F(x) - F(c)| = \left|\int_c^x f(t)dt\right| \leq \int_c^x |f(t)|dt \leq A(x - c) \quad \text{[by (2)]}$$

and for $x < c$, $|F(x) - F(c)| = \left|\int_c^x f(t)dt\right| = \left|\int_x^c f(t)dt\right| \leq A(c - x)$ [by (2)]

Hence, $|F(x) - F(c)| \leq A\,|x - c|$ for all $x \in [a, b]$ ... (3)

Let ε be any positive number. Then for this chosen ε, there exists a positive $\delta = \dfrac{\varepsilon}{A}$ such that

$|F(x) - F(c)| < \varepsilon$ whenever $|x - c| < \delta$ [from (3)]

$\Rightarrow$ F is continuous at c.

Since c is arbitrary point of $[a, b]$, F is continuous on $[a, b]$.

Remark : Since δ is indepedent of c, so we can conclude that F is uniformly continuous on $[a, b]$.

(ii) Let $P = \{x_0, x_1, x_2, ..., x_n\} \in \mathcal{P}[a, b]$.

For every $r = 1, 2, ... n$

$$|F(x_r) - F(x_{r-1})| = \left|\int_{x_{r-1}}^{x_r} f(t)dt\right| \leq \int_{x_{r-1}}^{x_r} |f(t)|dt < A(x_r - x_{r-1}).$$

Thus $V(P, F) = \sum_{r=1}^{n} |F(x_r) - F(x_{r-1})| < A(b - a)$ for every $P \in \mathcal{P}[a, b]$

$\Rightarrow$ Sup$\{V(P, F) : P \in \mathcal{P}[a, b]\} \leq A(b - a)$

$\Rightarrow$ total variation of F on $[a, b]$ is bounded above which proves that F is of bounded variation on $[a, b]$. (See the chapter on bounded variation.)

Theorem 5.9.2. (Indefinite Integral Theorem)

Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and $F(x) = \int_a^x f(t)dt$, $x \in [a, b]$. Then F is differentiable on $[a, b]$ and $\dfrac{d}{dx} F(x) = f(x)$, $x \in [a, b]$.

Proof. Let $c \in [a, b]$. Since f is continuous at c, for any positive ε, there exists a positive δ such that $|f(t) - f(c)| < \dfrac{\varepsilon}{2}$ when $|t - c| < \delta$ and $t \in [a, b]$ ...(1).

Let h be any real number such that $0 < |h| < \delta$ and $c + h \in [a, b]$.

Then

$$F(c + h) - F(c) = \int_a^{c+h} f(t)dt - \int_a^c f(t)dt = \int_c^{c+h} f(t)dt$$

$$\Rightarrow \frac{F(c+h) - F(c)}{h} - f(c) = \frac{1}{h}\int_c^{c+h} (f(t) - f(c))dt.$$

If $h > 0$ then $\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| \leq \frac{1}{h}\int_c^{c+h} |f(t) - f(c)|dt \leq \frac{\varepsilon}{2} < \varepsilon$

[by (1)]

and if $h < 0$ then $\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| \leq \frac{1}{-h}\int_{c+h}^{c} |f(t) - f(c)|dt$

$\leq \frac{\varepsilon}{2} < \varepsilon$ [by (1)]

Hence for $0 < |h| < \delta$, $c + h \in [a, b]$

$$\left|\frac{F(c+h) - F(c)}{h} - f(c)\right| < \varepsilon \Rightarrow \lim_{h \to 0} \frac{F(c+h) - F(c)}{h} = f(c)$$

$\Rightarrow F'(c)$ exists and $= f(c)$.

As c is any point of $[a, b]$, it is proved that $F'(x)$ exists for all $x \in [a, b]$ and $F'(x) = f(x)$, $x \in [a, b]$.

Theorem 5.9.3. (Fundamental Theorem of Integral Calculus)
If $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$ and φ is a primitive of f on $[a, b]$ *i.e.* $\varphi'(x) = f(x)$ for all $x \in [a, b]$ then $\int_a^b f(x)\, dx = \varphi(b) - \varphi(a)$.

Proof. Let $P = \{x_0, x_1, x_2, ..., x_n\}$ be a partition of $[a, b]$ and $I_r = [x_{r-1}, x_r]$ $(r = 1, 2, ..., n)$ be n sub intervals of $[a, b]$. Since f is integrable on $[a, b]$, it is bounded on $[a, b]$.

Let $M_r = \text{Sup}\{f(x) : x \in I_r\}$, $m_r = \text{Inf}\{f(x) : x \in I_r\}$ $(r = 1, 2, ..., n)$.

Since φ is a primitive of f on $[a, b]$, φ is continuous and differentiable on $[a, b]$ and $\varphi'(x) = f(x)$, $x \in [a, b]$.

Then by Lagrange's Mean Value Theorem applied on φ in $[x_{r-1}, x_r]$ for $r = 1, 2, ..., n$ we have

$\varphi(x_r) - \varphi(x_{r-1}) = \varphi'(\xi_r)(x_r - x_{r-1})$ for some $\xi_r \in (x_{r-1}, x_r)$

$= f(\xi_r)\,|I_r|\ (r = 1, 2, ..., n).$

Since $m_r \leq f(\xi_r) \leq M_r\ (r = 1, 2, ..., n)$

hence $L(P, f) \leq \sum_{r=1}^{n} f(\xi_r)\,|I_r| \leq U(P, f)$

$\Rightarrow L(P, f) \leq \sum_{r=1}^{n} [\varphi(x_r) - \varphi(x_{r-1})] \leq U(P, f)$

$\Rightarrow L(P, f) \leq \varphi(b) - \varphi(a) \leq U(P, f).$

Taking limit $\|P\| \to 0$ and using Darboux's Theorem we have

$$\underline{\int_a^b} f(x)dx \leq \varphi(b) - \varphi(a) \leq \overline{\int_a^b} f(x)dx.$$

Since f is integrable on $[a, b]$, $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \int_a^b f(x)dx.$

Therefore $\int_a^b f(x)dx = \varphi(b) - \varphi(a).$

> **Remark :** Evaluation of $\int_a^b f(x)dx$ in terms of primitive of f on $[a, b]$ is possible only if f is integrable on $[a, b]$ and f has no points of jump discontinuity in $[a, b]$.

***Example* 5.9.6.** Examine whether Fundamental Theorem of Integral Calculus is applicable to evaluate the integral $\int_0^3 f(x)dx$ where $f(x) = x[x]$, $x \in [0, 3]$.

***Solution* :** Explicit expression of f is

$$\begin{aligned} f(x) &= 0, & 0 \leq x < 1 \\ &= x, & 1 \leq x < 2 \\ &= 2x, & 2 \leq x < 3 \\ &= 9, & x = 3 \end{aligned}$$

f has jump discontinuities at $x = 1$, $x = 2$, $x = 3$. If φ is a primitive of f on $[0, 3]$ then $\varphi'(x) = f(x)$, $x \in [0, 3] \Rightarrow \varphi'$ has jump discontinuities at $x = 1$, $x = 2$ and $x = 3$ which contradicts the well known result that a derived function cannot have jump discontinuity in its domain. So f has no primitive on $[0, 3]$. Hence Fundamental Theorem of Integral Calculus cannot be applied to evaluate $\int_0^3 f(x)dx$.

***Example* 5.9.7.** Show ... Fundamental Theorem of Integral

5.10. ALTERNATIVE DEFINITION OF RIEMANN INTEGRAL.

Definition : Let $f : [a, b] \to \mathbb{R}$ be a function and $P = \{x_0, x_1, x_2, ..., x_n\}$ be a partition of $[a, b]$.

Let $I_r = [x_{r-1}, x_r]$ and $|I_r| = x_r - x_{r-1}$. If $\xi_r \in I_r$, $\Gamma = \{\xi_1, \xi_2, ..., \xi_n\}$ is collection of intermediate points $\xi_r (r = 1, 2, ..., n)$. Then the sum $\sum_{r=1}^{n} f(\xi_r) |I_r|$ is a real number depending on P and Γ which is denoted by $S(P, \Gamma, f)$.

$S(P, \Gamma, f) = \sum_{r=1}^{n} f(\xi_r) |I_r|$ is defined as Riemann sum of f for the partition P of $[a, b]$ and collection Γ of intermediate points ξ_r $(r = 1, 2, ..., n)$ in $[a, b]$. ξ_r's $(r = 1, 2, ..., n)$ are also called tags of I_r $(r = 1, 2, ..., n)$.

Note : Γ is a collection and not necessarily a set, since choice of tags may be such that $\xi_1, \xi_2, ..., \xi_n$ are not always distinct.

Definition : A function $f : [a, b] \to \mathbb{R}$ is said to be Riemann Integrable on $[a, b]$ if there exists a real number 'l' such that for every choice of positive number ε there exists a positive number δ such that

$$|S(P, \Gamma, f) - l| < \varepsilon$$

for all partitions $P \in \mathcal{P}[a, b]$ with $||P|| < \delta$, where $S(P, \Gamma, f)$ is the Riemann sum of f for a partition P of $[a, b]$ and for any choice of intermediate points of $[a, b]$ included in Γ.

Thus f is Riemann Integrable on $[a, b]$ if $\lim_{||P|| \to 0} S(P, \Gamma, f)$ exists (finitely)

and then $l = \int_a^b f(x)dx = \lim_{||P|| \to 0} S(P, \Gamma, f)$.

Remark : For any real number given by ||P|| there exist infinitely many partitions P of $[a, b]$ and hence $S(P, \Gamma, f)$ is not a single valued function of ||P||. So the above concept of limit is not covered by the limit defined in ordinary sense.

In fact $S(\, , f)$ is a real valued function whose domain is the set $\bigcup_{P \in \mathcal{P}[a,b]} \{(P, \Gamma) : \Gamma \text{ corresponds to given } P\}$.

The following theorem establishes the equivalence of two definitions of $\int_a^b f(x)dx$.

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Theorem 5.10.1.

If $f : [a, b] \to \mathbb{R}$ be a bounded function such that

$$\underline{\int}_a^b f(x)dx = \overline{\int}_a^b f(x)dx \left(= \int_a^b f(x)dx\right)$$

then $\lim_{\|P\|\to 0} S(P, \Gamma, f)$ exists and $= \int_a^b f(x)dx$ and conversely.

where $S(P, \Gamma, f)$ is the Riemann sum of f corresponding to partition P of $[a, b]$ and Γ, the collection of intermediate points of $[a, b]$ corresponding to P.

Proof. Let $\underline{\int}_a^b f(x)dx = \overline{\int}_a^b f(x)dx \left(= \int_a^b f(x)dx\right)$ ... (1)

Let $P = \{x_0, x_1, x_2, \ldots, x_n\}$ be any partition of $[a, b]$,

$I_r = [x_{r-1}, x_r]$ and $|I_r| = x_r - x_{r-1}$ $(r = 1, 2, \ldots, n)$.

Let $\Gamma = \{\xi_1, \xi_2, \ldots, \xi_n\}$ where $\xi_r \in I_r$ $(r = 1, 2, \ldots, n)$.

Riemann sum function for f and $P \in \mathcal{P}[a, b]$ is

$$S(P, \Gamma, f) = \sum_{r=1}^{n} f(\xi_r)\,|I_r|.$$

Let $M_r = \text{Sup}\{f(x) : x \in I_r\}$, $m_r = \text{Inf}\{f(x) : x \in I_r\}$ $(r = 1, 2, \ldots, n)$.

Now $m_r \le f(\xi_r) \le M_r$ $(r = 1, 2, \ldots n)$

$\Rightarrow L(P, f) \le S(P, \Gamma, f) \le U(P, f)$ for every $P \in \mathcal{P}[a, b]$.

By Darboux's Theorem, taking limit $\|P\| \to 0$ we have

$$\underline{\int}_a^b f(x)dx = \lim_{\|P\|\to 0} L(P, f) \le \lim_{\|P\|\to 0} S(P, \Gamma, f) \le \lim_{\|P\|\to 0} U(P, f) = \overline{\int}_a^b f(x)dx$$

$$\Rightarrow \int_a^b f(x)dx \le \lim_{\|P\|\to 0} S(P, \Gamma, f) \le \int_a^b f(x)dx \quad \text{(by (1))}$$

$$\Rightarrow \lim_{\|P\|\to 0} S(P, \Gamma, f) \text{ exists and } = \int_a^b f(x)dx.$$

Conversely, let $\lim_{\|P\|\to 0} S(P, \Gamma, f)$ exist and $= l$.

Let $P = \{x_0, x_1, \ldots, x_n\} \in \mathcal{P}[a, b]$, $I_r = [x_{r-1}, x_r]$,

$|I_r| = x_r - x_{r-1}$; M_r and m_r are respectively Supremum and Infimum of f on I_r $(r = 1, 2, \ldots, n)$.

From definition of Supremum, for any chosen positive number ε there exists a point $\xi'_r \in I_r$ such that

$$M_r - \frac{\varepsilon}{b-a} < f(\xi'_r) \le M_r \qquad (r = 1, 2, \ldots, n)$$

$$\Rightarrow 0 \le M_r - f(\xi'_r) < \frac{\varepsilon}{b-a} \qquad (r = 1, 2, \ldots, n).$$

If $\Gamma' = \{\xi_1', \xi_2', \ldots, \xi_n'\}$ then we have

$$0 \le \sum_{r=1}^{n} [M_r - f(\xi_r')]\, |I_r| < \varepsilon$$

$$\Rightarrow \quad 0 \le U(P, f) - S(P, \Gamma', f) < \varepsilon. \qquad \ldots (2)$$

Taking limit $||P|| \to 0$ in the inequality (2) and using Darboux's theorem we have

$$0 \le \overline{\int_a^b} f(x)dx - l \le \varepsilon.$$

Since ε is arbitrary, we have $\overline{\int_a^b} f(x)dx = l.$

From definition of Infimum, for any chosen positive number ε there exists a point $\xi_r'' \in I_r$ such that

$$m_r \le f(\xi_r'') < m_r + \frac{\varepsilon}{b-a} \qquad (r = 1, 2, \ldots, n)$$

$$\Rightarrow \quad 0 \le f(\xi_r'') - m_r < \frac{\varepsilon}{b-a} \qquad (r = 1, 2, \ldots, n)$$

$$\Rightarrow \quad 0 \le S(P, \Gamma'', f) - L(P, f) < \varepsilon \qquad \ldots (3)$$

where $\Gamma'' = \{\xi_1'', \xi_2'', \ldots, \xi_n''\}$.

Taking limit $||P|| \to 0$ in the equality (3) and using Darboux's theorem we have

$$0 \le l - \underline{\int_a^b} f(x)dx \le \varepsilon.$$

Since ε is arbitrary, $\underline{\int_a^b} f(x)dx = l.$

Hence it is proved that $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx.$

This establishes equivalence of two definitions of Riemann Integral.

Note : If $f \in \mathcal{R}[a, b]$ then $\lim\limits_{||P|| \to 0} S(P, \Gamma, f)$ exists uniquely.

Theorem 5.10.2.
Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $\{P_n\}_n$ be a sequence of partitions of $[a, b]$ such that the sequence $\{||P_n||\}_n$ converges to 0. Then f is integrable on $[a, b]$ if and only if $\lim\limits_{n \to \infty} S(P_n, \Gamma, f)$ exists and is equal to $\int_a^b f(x)dx$, where $S(P_n, \Gamma, f)$ is the Riemann sum of f for $P_n \in \mathcal{P}[a, b]$ and collection Γ of intermediate points of $[a, b]$ corresponding to P.

Proof. Let f be integrable on $[a, b]$. Then for any chosen positive number ε there exists a positive number δ such that

$$U(P, f) - L(P, f) < \varepsilon$$

for every partition P of $[a, b]$ satisfying $||P|| < \delta$.

Since $\{P_n\}_n$ is a sequence of partitions of $[a, b]$ such that the sequence $\{||P_n||\}_n$ converges to 0, so there exists a natural number m (depending on δ) such that $||P_n|| < \delta$ for all $n \geq m$.

So $U(P_n, f) - L(P_n, f) < \varepsilon$ for all $n \geq m$.

For every $P_n \in \mathcal{P}[a, b]$ we have

$$L(P_n, f) \leq S(P_n, \Gamma, f) \leq U(P_n, f) \quad \text{... (1)}$$

where $S(P_n, \Gamma, f)$ is the Riemann sum function of f.

Since f is integrable on $[a, b]$, for every $P_n \in \mathcal{P}[a, b]$

$$L(P_n, f) \leq \int_a^b f(x)dx \leq U(P_n, f) \quad \text{... (2)}$$

From (1) and (2), we have

$$\left|S(P_n, \Gamma, f) - \int_a^b f(x)dx\right| \leq U(P_n, f) - L(P_n, f) < \varepsilon \text{ for all } n \geq m$$

$$\Rightarrow \lim_{n\to\infty} S(P_n, \Gamma, f) \text{ exists and } = \int_a^b f(x)dx.$$

Conversely, let $\lim_{n\to\infty} S(P_n, \Gamma, f)$ exists and $= l$

for every partition P_n of $[a, b]$ such that $||P_n|| \to 0$ as $n \to \infty$.

Then for any positive number ε there exists a natural number m such that

$$|S(P_n, \Gamma, f) - l| < \frac{\varepsilon}{4} \text{ for all } n \geq m.$$

Let $||P_m|| = \delta' > 0$. Since $\lim_{n\to\infty} ||P_n|| = 0$, there exists a natural number k such that $||P_n|| < \delta'$ for all $n \geq k$. Let $q = \max\{m, k\}$.

Then $|S(P_n, \Gamma, f) - l| < \frac{\varepsilon}{4}$ for all $n \geq q$.

This implies that $|S(P_n, \Gamma, f) - l| < \frac{\varepsilon}{4}$ for all partitions P_n of $[a, b]$ satisfying $||P_n|| < \delta'$.

Let $P' \in \mathcal{P}[a, b]$ such that $\|P'\| < \delta'$.

Let $P' = \{x_0, x_1, ..., x_n\}$ and $I_r = [x_{r-1}, x_r]$, $|I_r| = x_r - x_{r-1}$ $(r = 1, 2, ..., n)$.

Let $M_r = \text{Sup}\{f(x) : x \in I_r\}$, $m_r = \text{Inf}\{f(x) : x \in I_r\}$ $(r = 1, 2, ..., n)$.

From properties of Supremum and Infimum of a bounded set, for the chosen ε, there exist points α_r, β_r in I_r such that

$$M_r - \frac{\varepsilon}{4(b-a)} < f(\alpha_r) \le M_r$$

and $$m_r \le f(\beta_r) < m_r + \frac{\varepsilon}{4(b-a)} \quad \ldots (3) \quad (r = 1, 2, \ldots n).$$

If $\Gamma_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, $\Gamma_2 = \{\beta_1, \beta_2, \ldots, \beta_n\}$ we have from (3)

$$U(P', f) - \frac{\varepsilon}{4} < S(P', \Gamma_1, f) \le U(P', f)$$

and $$L(P', f) \le S(P', \Gamma_2, f) < L(P', f) + \frac{\varepsilon}{4}$$

Since $l - \frac{\varepsilon}{4} < S(P_n, \Gamma, f) < l + \frac{\varepsilon}{4}$ for any choice of intermediate points in Γ and $||P_n|| < \delta'$, we have therefore

$$U(P', f) - \frac{\varepsilon}{4} < l + \frac{\varepsilon}{4} \text{ and } L(P', f) + \frac{\varepsilon}{4} > l - \frac{\varepsilon}{4}$$

$$\Rightarrow l - \frac{\varepsilon}{2} < L(P', f) \le U(P', f) < l + \frac{\varepsilon}{2} \quad \ldots (4)$$

$\Rightarrow U(P', f) - L(P', f) < \varepsilon$ for any partition P' of $[a, b]$ satisfying $||P'|| < \delta'$, a sufficient condition of integrability.

Hence f is integrable on $[a, b]$.

Also $L(P', f) \le \int_a^b f(x)dx \le U(P', f)$, so using(4),

$$l - \frac{\varepsilon}{2} < \int_a^b f(x)dx < l + \frac{\varepsilon}{2}$$

$\Rightarrow \left|\int_a^b f(x)dx - l\right| < \frac{\varepsilon}{2}$ which holds for every positive ε.

Hence $l = \int_a^b f(x)dx$.

This completes the proof.

Note. 1. If $\lim\limits_{n\to\infty} S(P_n, \Gamma_1, f) = l_1$ and $\lim\limits_{n\to\infty} S(P_n, \Gamma_2, f) = l_2$, for different choices of intermediate points in Γ_1 and Γ_2 and $l_1 \ne l_2$ then $\lim\limits_{n\to\infty} S(P_n, \Gamma, f)$ does not exist and hence f is not integrable on $[a, b]$.

2. If $P_n = \{x_0, x_1, \ldots, x_n\} \in \mathcal{P}[a, b]$, in particular we can take the points of P_n equispaced *i.e.*, $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1}$ as $n \to \infty$. Then if for any choice of intermediate points $\xi_1, \xi_2, \ldots, \xi_n$ in Γ

$\lim\limits_{n\to\infty} S(P_n, \Gamma, f) = \lim\limits_{n\to\infty} \frac{b-a}{n} \sum\limits_{i=1}^{n} f(\xi_i)$ exists and is equal to l then f