Central force:

A force ($\vec{F}$) acting on a particle is said to be central if

(i) it is always directed towards or away from a fixed point i.e. is along the position vector $\vec{r}$.

(ii) its magnitude is a function of the distance (r) of the particle from the fixed point. i.e. $F = f(r)$

Mathematically, $\vec{F} = f(r)\,\hat{r}$

where, $\hat{r}$ = unit vector along $\vec{r}$.

Characteristics:

The characteristics of central force between two bodies are

(i) central forces are long range forces i.e. they are effective even when the distance between the interacting bodies is very large. The magnitude of the force depends only on the distance between the centres of the two bodies.

(ii) It acts along the line joining the centres of the two bodies. If $\vec{r}$ is a vector joining the centres of the two bodies and $\hat{r}$ is a unit vector along $\vec{r}$, then a central force can always be represented as $\vec{F} = f(r)\,\hat{r}$.

(iii) A central force is a conservative force because the work done under this force is independent of the path.
The ~~cute~~ curl of a central force is zero. Mathematically, $\vec{\nabla} \times \vec{F} = 0$
A central force is the gradient of some scalar function V
i.e. $\vec{F} = -\,\text{grad}\,V = -\vec{\nabla} V$.

Examples :

(i) Gravitational force of attraction between two masses.

(ii) Electrostatic force (or Coulomb force) of attraction or repulsion between two charges.

Show that a two body central force problem can be reduced to a single body central force problem.

Let us consider two particles of masses m_1 and m_2 separated by a distance r and acted on by external forces $\vec{F}_1^{ext}$ and $\vec{F}_2^{ext}$ and internal forces $\vec{F}_{12}^{int}$, $\vec{F}_{21}^{int}$ respectively.

The internal forces satisfy Newton's third law given by

$$\vec{F}_{12}^{int} = -\vec{F}_{21}^{int}$$

The equation of motion for the two particles are

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = \vec{F}_1^{ext} + \vec{F}_{12}^{int} \quad \text{—— (i)}$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = \vec{F}_2^{ext} + \vec{F}_{21}^{int} \quad \text{—— (ii)}$$

Multiplying (i) by m_2 and (ii) by m_1 and then subtracting we get,

$$m_1 m_2 \frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = \left(m_2 \vec{F}_1^{ext} - m_1 \vec{F}_2^{ext}\right) + \left(m_2 \vec{F}_{12}^{int} - m_1 \vec{F}_{21}^{int}\right)$$

or, $$m_1 m_2 \frac{d^2(\vec{r}_1 - \vec{r}_2)}{dt^2} = \left(m_2 \vec{F}_1^{ext} - m_1 \vec{F}_2^{ext}\right) + (m_1 + m_2)\vec{F}_{12}^{int}$$

or, $$m_1 m_2 \frac{d^2}{dt^2}(\vec{r}_1 - \vec{r}_2) = m_1 m_2 \left(\frac{\vec{F}_1^{ext}}{m_1} - \frac{\vec{F}_2^{ext}}{m_2}\right) + (m_1 + m_2)\vec{F}_{12}^{int} \quad \text{—(iii)}$$

Dividing equation (iii) through out by m_1+m_2 and defining the reduced mass of the system by the formula,

$$\mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\therefore \quad \mu \frac{d^2}{dt^2}\left(\vec{r}_1 - \vec{r}_2\right) = \vec{F}_{12}^{\,int} + \mu\left(\frac{\vec{F}_1^{\,ext}}{m_1} - \frac{\vec{F}_2^{\,ext}}{m_2}\right) \quad \text{—(iv)}$$

(i) If no external force is acting

$$\vec{F}_1^{\,ext} = \vec{F}_2^{\,ext} = 0$$

or, (ii) if external forces $\vec{F}_1^{\,ext}$ and $\vec{F}_2^{\,ext}$ are proportional to the masses of the particles on which they act and produce equal acceleration in the two particles, i.e. if

$$\frac{\vec{F}_1^{\,ext}}{m_1} = \frac{\vec{F}_2^{\,ext}}{m_2}$$

equation (iv) reduces to

$$\mu \frac{d^2\vec{r}}{dt^2} = \vec{F}_{12}^{\,int} \qquad \left[\because \vec{r} = \vec{r}_1 - \vec{r}_2\right]$$

Here $\vec{r}$ denotes the relative co-ordinate of m_1 w.r.to m_2.

This is the equation of motion of a particle having mass equal to reduced mass μ and moving under the action of force $\vec{F}_{12}^{\,int}$ (which the particle of mass m_2 exerts on the particle of mass m_1)

Thus two body problem can be considered as a single body problem.

Impratant features of motion under central force field :

1. The angular momentum of the particle is conserved.

The angular momentum of a particle under a central force always remains constant.

Let us consider the motion of a particle of mass m in a central force field. Let (r, θ) be the polar co-ordinate of the particle at any instant.

The radial and transverse acceleration of the particle are given by

$$f_r = \ddot{r} - r\dot{\theta}^2$$

$$f_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

Hence, the equations of motion along r and θ directions are

$$m\ddot{r} - mr\dot{\theta}^2 = F(r) \quad \text{———(i)}$$

and, $$mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = F_\theta$$

$$= 0 \quad \text{———(2)} \quad [\because F_\theta = 0 \text{ in a central force field}]$$

Equation (2) can be written as

$$\frac{d}{dt}(mr^2\dot{\theta}) = 0$$

or, $mr^2\dot{\theta} = \text{Constant}$

or, $L = \text{Constant}$. $[\because mr^2\dot{\theta} = L$, the angular momentum$]$

ie. in a central force field the angular momentum of a particle is conserved.

2. The orbit of the particle moving under the action of a central force, lies in a plane.

Physical Explanation :

In such cases the force vector, and hence the acceleration, are parallel to the radius vector and so velocity, acceleration and radius vector lie in a plane. The particle describing the motion can never leave this plane since there is no component of the velocity out of this plane.

Mathematical Explanation :

Any central force can be expressed as

$$\vec{F} = f(r)\,\hat{r}$$

$$\therefore \quad \vec{r} \times \vec{F} = f(r)\,\vec{r} \times \hat{r} = 0 \quad \text{--- (i)} \quad [\because \hat{r} = \text{unit vector along } \vec{r}]$$

$$\text{or,} \quad \vec{r} \times m\ddot{\vec{r}} = 0 \quad [\because \vec{F} = m\ddot{\vec{r}}]$$

$$\text{or,} \quad m(\vec{r} \times \ddot{\vec{r}}) = 0$$

$$\text{or,} \quad \vec{r} \times \frac{d}{dt}(\dot{\vec{r}}) = 0$$

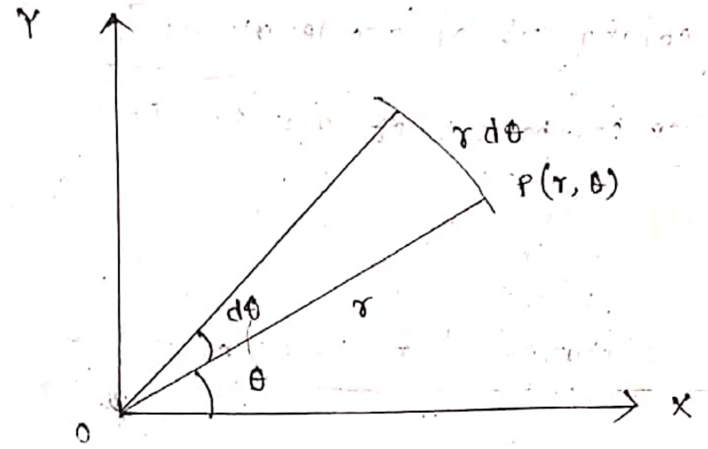
$$\text{or,} \quad \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) - \dot{\vec{r}} \times \dot{\vec{r}} = 0 \quad \left[\because \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = \vec{r} \times \frac{d\dot{\vec{r}}}{dt} + \dot{\vec{r}} \times \dot{\vec{r}}\right]$$

$$\text{or,} \quad \frac{d}{dt}(\vec{r} \times \dot{\vec{r}}) = 0$$

$$\text{or,} \quad \vec{r} \times \dot{\vec{r}} = \vec{h}$$

where $\vec{h}$ is a constant vector, i.e. independent of time and is normal to the plane containing $\vec{r}$ and $\dot{\vec{r}}$ and so the motion takes place in a plane.

3. The areal velocity of a particle in a central force field is constant :



Let, the area swept out by the radius vector $(\vec{OP})$ in time dt is dA,

$$\therefore\ dA = \frac{1}{2} r (r\, d\theta)$$

$$= \frac{1}{2} r^2\, d\theta$$

Hence, the area swept per unit time, i.e. areal velocity

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$= \frac{1}{2} r^2 \dot{\theta}$$

$$= \frac{1}{2m} (m r^2 \dot{\theta})$$

$$= \frac{L}{2m} = \text{constant}.$$

Q: Show that the sum of the Kinetic and potential energies of a body driven by a central force, having no explicit time ~~dependence~~ depends remain constant. (B.U- 2005)

In a central force field we have,

angular momentum $L = mr^2\dot{\theta}$ = Constant —(i)

If $F(r)$ be the central force, the equation of motion of a body of mass m under central force is given by

$$m(\ddot{r} - r\dot{\theta}^2) = F(r)$$

$$\text{or, } m\left(\ddot{r} - r\frac{L^2}{m^2r^4}\right) = F(r)$$

$$\text{or, } m\left(\ddot{r} - \frac{L^2}{m^2r^3}\right) = F(r) \quad \ldots\ldots (ii)$$

Since a central force is conservative, it can be ~~driv~~ derived from a potential v which is a function of r alone.

$$\text{i.e. } F(r) = -\frac{\partial v}{\partial r} \quad \ldots\ldots (iii)$$

∴ Equation (ii) becomes,

$$m\ddot{r} = \frac{L^2}{m^2r^3} - \frac{\partial v}{\partial r}$$

$$= -\frac{\partial}{\partial r}\left[\frac{1}{2}\left(\frac{L^2}{mr^2}\right) + v\right]$$

Multiplying both side of the equation by $\dot{r}$ we have

$$m\dot{r}\ddot{r} = -\dot{r}\frac{\partial}{\partial r}\left[\frac{1}{2}\left(\frac{L^2}{mr^2}\right) + v\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{r}^2\right) = -\frac{d}{dt}\left[\frac{1}{2}\left(\frac{L^2}{mr^2}\right) + v\right]$$

or, $\frac{d}{dt}\left[\frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{m r^2} + V \right] = 0$

$\therefore \frac{1}{2} m \dot{r}^2 + \frac{1}{2} \frac{L^2}{m r^2} + V = \text{Constant}$

$\Rightarrow \frac{m}{2} \left(\dot{r}^2 + \frac{m^2 r^4 \dot{\theta}^2}{m^2 r^2} \right) + V = \text{Constant}$

$\Rightarrow \frac{m}{2} \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + V = \text{Constant}$

$\Rightarrow \frac{1}{2} m v^2 + V = \text{Constant}$ $\left[\because \vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} \right.$
$\left. v^2 = \dot{r}^2 + r^2 \dot{\theta}^2 \right]$

i.e. K.E + P.E = Constant.

$\therefore$ The sum of the kinetic and potential energies of a body driven by a central force, having no explicite time depends remain Constant.

- **Q:** Prove that the total energy of a particle of mass m acted upon by a central force is given by

$$E = \frac{h^2}{2m} \left[u^2 + \left(\frac{du}{d\theta}\right)^2 \right] + v(r)$$

where $v(r)$ is the potential energy and h is the angular momentum of the body; $u = \frac{1}{r}$, r and θ being the polar coordinates of the particle.

Kinetic energy in $(r-\theta)$ plane $= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2)$

Hence the total energy is given by

$$E = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2), + v(r)$$

where $v(r)$ is the potential energy.

Now the angular momentum is $mr^2\dot{\theta} = h$

$$\therefore r^2\dot{\theta}^2 = \frac{h^2}{m^2r^2} = \frac{h^2u^2}{m^2} \quad \left[\text{putting } \frac{1}{r^2} = u^2\right]$$

Again,

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right)$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\frac{h}{m}u^2$$

$$= -\frac{h}{m}\frac{du}{d\theta}$$

Substituting the values of $r^2\dot{\theta}^2$ and $\dot{r}^2$ in the expression for total energy we have

$$E = \frac{1}{2}m\left[\frac{h^2}{m^2}\left(\frac{du}{d\theta}\right)^2 + \frac{h^2}{m^2}u^2\right] + V(r)$$

$$= \frac{h^2}{2m}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] + V(r)$$

- **Q: Show that all central forces are conservative.**

For central force $\vec{F}(r) = F(r)\,\hat{r}$

$$= \frac{F(r)}{r}\vec{r} \quad [\because \vec{r} = r\hat{r}]$$

$$= \frac{F(r)}{r}(x\hat{i} + y\hat{j} + z\hat{k}) \quad [\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}]$$

$$= F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

where, $F_x = \frac{F(r)}{r}x$, $F_y = \frac{F(r)}{r}y$, $F_z = \frac{F(r)}{r}z$

Now, $\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \hat{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \hat{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)$$

$$\therefore \left(\vec{\nabla} \times \vec{F}\right)_x = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}$$

Now, $\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\left[\frac{z}{r} F(r)\right]$

$$= z \frac{d}{dr}\left[\frac{F(r)}{r}\right] \frac{\partial r}{\partial y}$$

$$= \frac{yz}{r} \frac{d}{dr}\left[\frac{F(r)}{r}\right] \quad \left[\because \frac{\partial r}{\partial y} = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{2y}{2\sqrt{x^2+y^2+z^2}} = \frac{y}{r}\right]$$

Similarly $\frac{\partial F_y}{\partial z} = \frac{yz}{r} \frac{d}{dr}\left[\frac{F(r)}{r}\right] = \frac{\partial F_z}{\partial y}$

$$\therefore \left(\vec{\nabla} \times \vec{F}\right)_x = 0$$

Similarly, $\left(\vec{\nabla} \times \vec{F}\right)_y = 0$

$\left(\vec{\nabla} \times \vec{F}\right)_z = 0$

Hence, $\vec{\nabla} \times \vec{F} = 0$

i.e. all central force are conservative.

- ## The general differential equation for central orbit :

The equation of motion of a ~~central~~ particle in a central force field is :

$$m\ddot{r} - mr\dot{\theta}^2 = f(r) \longrightarrow (I)$$

$$\text{and } mr\ddot{\theta} + 2m\dot{r}\dot{\theta} = 0 \longrightarrow (II)$$

where,
$f(r)$ is the central force along the radius vector $\vec{r}$.

Angular momentum, $L = mr^2\dot{\theta}$

$$\text{or, } \dot{\theta} = \frac{L}{mr^2}$$

So Eq^n (II) can be written as

$$m\ddot{r} - mr\frac{L^2}{m^2r^4} = f(r)$$

$$\text{or, } m\ddot{r} - \frac{L^2}{mr^3} = f(r) \longrightarrow (III)$$

Let us put, $u = \frac{1}{r}$

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right)$$

$$= -\frac{1}{u^2}\cdot\frac{du}{d\theta}\,\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\cdot\frac{du}{d\theta}\cdot\frac{L}{mr^2} \quad \left[\because \dot{\theta} = \frac{L}{mr^2}\right]$$

$$= -\frac{1}{u^2}\cdot\frac{Lu^2}{m}\,\frac{du}{d\theta} \quad \left[\because \frac{1}{u} = r\right]$$

$$= -\frac{L}{m}\,\frac{du}{d\theta}$$

Twice the areal velocity $h = r^2\dot{\theta}$

$$\text{or, } \dot{\theta} = h/r^2$$

So, Eq^n (II) can be written as

$$m\ddot{r} - mr.\frac{h^2}{r^4} = f(r)$$

$$\text{or, } m\ddot{r} - m\frac{h^2}{r^3} = f(r) \longrightarrow (III)$$

Let us put, $u = \frac{1}{r}$

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right)$$

$$= -\frac{1}{u^2}\,\frac{du}{d\theta}\,\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\cdot\frac{h}{r^2}\,\frac{du}{d\theta} \quad \left[\because \dot{\theta} = h/r^2\right]$$

$$= -\frac{u^2}{u^2}\,h\,\frac{du}{d\theta}$$

$$= -h\,\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{dt}\left(-\frac{L}{m}\frac{du}{d\theta}\right)$$

$$= -\frac{L}{m}\frac{d^2u}{d\theta^2}\dot{\theta}$$

$$= -\frac{L}{m}\frac{Lu^2}{m}\frac{d^2u}{d\theta^2}$$

$$= -\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}$$

$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

$$= -h\frac{d^2u}{d\theta^2}\dot{\theta}$$

$$= -h(hu^2)\frac{d^2u}{d\theta^2}$$

$$= -h^2u^2\frac{d^2u}{d\theta^2}$$

substituting the values of r and $\ddot{r}$ in eqn (III) we get,

$$m\left(-\frac{L^2u^2}{m^2}\frac{d^2u}{d\theta^2}\right) - \frac{L^2u^3}{m} = F(1/u)$$

$$\Rightarrow -\frac{L^2u^2}{m}\left(\frac{d^2u}{d\theta^2} + u\right) = F(1/u)$$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = -\frac{m\,F(1/u)}{L^2u^2} \longrightarrow (IV)$$

$$m\left(-h^2u^2\frac{d^2u}{d\theta^2}\right) - mh^2u^3 = F(1/u)$$

$$\Rightarrow -mh^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = F(1/u)$$

$$\frac{d^2u}{d\theta^2} + u = -\frac{F(1/u)}{mh^2u^2} \longrightarrow (IV)$$

If the central force varies as the inverse square law,

then $F(r) = -K/r^2$

or, $F(1/u) = -Ku^2$

$$\frac{d^2u}{d\theta^2} + u = \frac{Km}{L^2} \longrightarrow (V)$$

$$\text{or, } \frac{d^2u}{d\theta^2} + \left(u - \frac{Km}{L^2}\right) = 0$$

$$\text{or, } \frac{d^2}{d\theta^2}\left(u - \frac{Km}{L^2}\right) + \left(u - \frac{Km}{L^2}\right) = 0$$

$$\frac{d^2u}{d\theta^2} + u = \frac{K}{mh^2} \longrightarrow (V)$$

$$\text{or, } \frac{d^2u}{d\theta^2} + \left(u - \frac{K}{mh^2}\right) = 0$$

$$\text{or, } \frac{d^2}{d\theta^2}\left(u - \frac{K}{mh^2}\right) + \left(u - \frac{K}{mh^2}\right) = 0$$

To find the sol^n of the above equation let us put

$$y = u - \frac{km}{L^2} \qquad\qquad y = u - \frac{k}{mh^2}$$

$$\therefore \frac{d^2y}{d\theta^2} + y = 0.$$

This is well known equation of harmonic oscillator of unit frequency. So, the general solution of the ~~given~~ above equation is given by

$$y = A\cos(\theta - \theta_0)$$

where, A and θ_0 are arbitrary constant.

$$\therefore u - \frac{km}{L^2} = A\cos(\theta - \theta_0)$$

$$\text{or, } \frac{1}{r} = \frac{mk}{L^2} + A\cos(\theta - \theta_0)$$

$$\text{or } \frac{1}{r} = \frac{mk}{L^2}\left\{1 + \frac{L^2 A}{km}\cos(\theta - \theta_0)\right\}$$

$$\text{or } \frac{(L^2/mk)}{r} = 1 + \frac{AL^2}{km}\cos(\theta - \theta_0) \longrightarrow (\text{VI})$$

$$u - \frac{k}{mh^2} = A\cos(\theta - \theta_0)$$

$$\text{or, } \frac{1}{r} = \frac{k}{mh^2} + A\cos(\theta - \theta_0)$$

$$\text{or, } \frac{1}{r} = \frac{k}{mh^2}\left\{1 + \frac{Amh^2}{k}\cos(\theta - \theta_0)\right\}$$

$$\text{or, } \frac{(mh^2/k)}{r} = 1 + \frac{Amh^2}{k}\cos(\theta - \theta_0) \longrightarrow (\text{VI})$$

It is always possible to choose the axes so that $\theta_0 = 0$.

Equation (VI) is similar to $\frac{l}{r} = 1 + \epsilon\cos\theta \longrightarrow (\text{VII})$

Eq^n (VII) is the equation of a conic whose semi-latus rectum is l and ϵ is the eccentricity of the conic.

Comparing eq^n (VI) and (VII) we get,

The semi latus rectum $l = \frac{L^2}{mk}$

and eccentricity $\epsilon = \frac{AL^2}{km}$

$$l = \frac{mh^2}{k}$$

$$\epsilon = \frac{Amh^2}{k}$$

Thus, under the inverse square law of attraction, the orbit will be a conic.

Total energy : $E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) + V(r)$

where, $V(r)$ = potential energy

$$V(r) = -\int F(r)\, dr = \int (K/r^2)\, dr = -\frac{K}{r} = -Ku$$

we have choosen the constant of integration so that $\lim_{r \to \infty} V = 0$.

$$\therefore E = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right) - Ku$$

$$u = \frac{mK}{L^2} + A \cos\theta$$

$$\cancel{E = \frac{1}{2} m \left[\frac{L^2 u^4}{m^2} + r^2 \frac{L^2}{m^2 r^4} \right] - ku}$$

$$\cancel{= \frac{A}{2}}$$

$$E = \frac{m}{2} \left[\frac{L^2}{m^2} \left(\frac{du}{d\theta} \right)^2 + \frac{1}{u^2} \cdot \frac{L^2 u^4}{m^2} \right] - ku$$

$$= \frac{m}{2} \cdot \frac{L^2}{m^2} \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right] - ku$$

$$\Rightarrow \left(\frac{du}{d\theta} \right)^2 + u^2 = (E + ku) \frac{2m}{L^2}$$

$$u = \frac{K}{mh^2} + A \cos\theta$$

$$E = \frac{m}{2} \left[h^2 \left(\frac{du}{d\theta} \right)^2 + \frac{1}{u^2} h^2 u^4 \right] - Ku$$

$$= \frac{mh^2}{2} \left[\left(\frac{du}{d\theta} \right)^2 + u^2 \right] - Ku$$

$$\left(\frac{du}{d\theta} \right)^2 + u^2 = \frac{2(E + Ku)}{mh^2}$$

$$(-A \sin\theta)^2 + \left[\frac{K}{mh^2} + A \cos\theta \right]^2 = \frac{2}{mh^2} \left[E + K \left(\frac{K}{mh^2} + A \cos\theta \right) \right]$$

$$\Rightarrow (-A \sin\theta)^2 + \left(\frac{mk}{L^2} + A \cos\theta \right)^2 = \left[E + k \left[\frac{mk}{L^2} + A \cos\theta \right] \right] \frac{2m}{L^2}$$

$$\Rightarrow A^2 \sin^2\theta + A^2 \cos^2\theta + \frac{m^2 k^2}{L^4} + \cancel{\frac{2 A mk}{L^2} \cos\theta} = \frac{2Em}{L^2} + \frac{2m^2 k^2}{L^4} + \cancel{\frac{2mk}{L^2} A \cos\theta}$$

$$\Rightarrow A^2 = \frac{2Em}{L^2} + \frac{m^2 k^2}{L^4}$$

$$A = \sqrt{\frac{2Em}{L^2} + \frac{m^2K^2}{L^4}}$$

$$= \frac{mK}{L^2}\sqrt{1 + \frac{2L^2E}{mK^2}}$$

$$\therefore\ u = \frac{1}{r} = \frac{mK}{L^2}\left[1 + \sqrt{1 + \frac{2L^2E}{mK^2}}\cos\theta\right]$$

$$A^2\sin^2\theta + A^2\cos^2\theta + \frac{K^2}{m^2h^4} + \frac{2K\cancel{A\cos\theta}}{\cancel{m}h^2}$$

$$= \frac{2E}{mh^2} + \frac{2K^2}{m^2h^4} + \frac{2K\cancel{A\cos\theta}}{\cancel{m}h^2}$$

$$A^2 = \frac{2E}{mh^2} + \frac{K^2}{m^2h^4}$$

$$A^2 = \frac{K^2}{m^2h^4}\left[1 + \frac{2Em^2h^4}{mh^2K^2}\right]$$

$$A = \frac{K}{mh^2}\sqrt{1 + \frac{2Emh^2}{K^2}}$$

Shape of the orbit:

$$u = \frac{K}{mh^2}\left[1 + \sqrt{1 + \frac{2Emh^2}{K^2}}\cos\theta\right]$$

The nature of the orbit is determined by the value of eccentricity

$$\varepsilon = \sqrt{1 + \frac{2L^2E}{mK^2}} \qquad \varepsilon = \sqrt{1 + \frac{2Emh^2}{K^2}}$$

Now, ε depends on the total energy E.

(I) If $E > 0$, $\varepsilon > 1$ and the orbit is hyperbola.

(II) If $E = 0$, $\varepsilon = 1$ and the orbit is parabola.

(III) If $E < 0$, $\varepsilon < 1$ and the orbit is ellipse.

(IV) If $E = -\frac{mK^2}{2L^2}$, $\varepsilon = 0$ and the orbit is a circle.

$$E = -\frac{K^2}{2mh^2}$$