Q. Consider a uniformly charged spherical volume of radius R which contains a total charge Q. Find the electric field and the electrostatic potential at all points in the space.

Soln: i) For $r \geq R$. $\rho = \dfrac{Q}{\frac{4}{3}\pi R^3}$

$$\oint \vec{E}_1 \cdot d\vec{s} = \frac{1}{\epsilon_0}\int \rho \, dV = \frac{1}{\epsilon_0} \cdot Q \quad \text{(From Gauss's Law)}$$

$$\Rightarrow E_1 \cdot 4\pi r^2 = \frac{Q}{\epsilon_0} \Rightarrow \boxed{\vec{E}_1 = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}}$$

$$\phi_1(r) = -\int_\infty^r \vec{E}_1 \cdot d\vec{r} = -\int_\infty^r \frac{Q}{4\pi\epsilon_0} \cdot \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0} \cdot \left[\frac{1}{r}\right]_\infty^r$$

$$\Rightarrow \boxed{\phi_1(r) = \frac{Q}{4\pi\epsilon_0 r}}$$

ii) For $r \leq R$

$$\oint \vec{E}_2 \cdot d\vec{s} = \frac{1}{\epsilon_0}\int \rho \, dv = \frac{1}{\epsilon_0}\int \left(\frac{Q}{\frac{4}{3}\pi R^3}\right) dv \quad \text{(From Gauss's law)}$$

$$\Rightarrow E_2 \cdot 4\pi r^2 = \frac{Q}{\frac{4}{3}\pi\epsilon_0 R^3} \cdot \frac{4}{3}\pi r^3 = \frac{Q r^3}{\epsilon_0 R^3}$$

$$\Rightarrow \boxed{\vec{E}_2 = \frac{Q\, r}{4\pi\epsilon_0 R^3}\hat{r}}$$

$$\therefore \phi_2(r) = -\int_\infty^r \vec{E} \cdot d\vec{r} = -\underset{r>R}{\int_\infty^R} \vec{E}_1 \cdot d\vec{r} - \underset{r\leq R}{\int_R^r} \vec{E}_2 \cdot d\vec{r}$$

$$= -\int_\infty^R \frac{Q}{4\pi\epsilon_0 r^2} dr - \int_R^r \left(\frac{Q}{4\pi\epsilon_0 R^3}\right) r \, dr$$

$\frac{1}{R} + \frac{1}{2R} = \frac{3}{2R}$

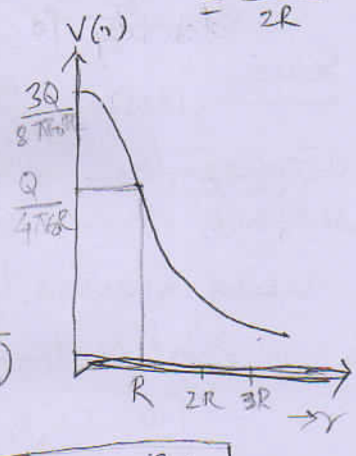
$$= \frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_\infty^R - \left(\frac{Q}{4\pi\epsilon_0 R^3}\right)\left[\frac{r^2}{2}\right]_R^r$$

$$= \frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{R} - \frac{Q}{4\pi\epsilon_0 R^3}\left[\frac{r^2}{2} - \frac{R^2}{2}\right]$$

$$= \frac{Q}{4\pi\epsilon_0} \cdot \frac{1}{R} - \frac{Q}{4\pi\epsilon_0 R^3} \cdot \frac{r^2}{2} + \frac{Q}{4\pi\epsilon_0 (2R)}$$

$$= \frac{Q}{4\pi\epsilon_0} \cdot \frac{3}{2R} - \frac{Q}{4\pi\epsilon_0 R^3} \cdot \frac{r^2}{2}$$

$$\boxed{\phi_2(r) = \frac{Q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right)}$$



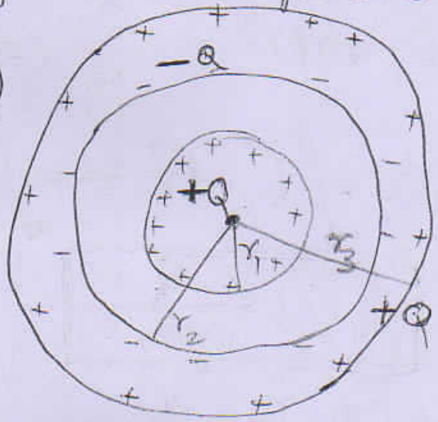
Q. A solid conducting sphere of radius r_1 has a charge of $+Q$. It is surrounded by concentric hollow conducting sphere of inside radius r_2 and outside radius r_3. Use Gauss's law to get expression for :

a) the field outside the outer sphere.

b) the field between the spheres.

c) Set up an expression for the potential of the inner sphere. (It is not necessary to perform the integrations).

Soln:



a) $\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$ (For $r > r_3$)

b) $\vec{E}(r) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$ (For $r_2 > r > r_1$)

c) Using the expression for the potential $\phi(P) = -\int_{\infty}^{P} \vec{E}\cdot d\vec{r}$,

we find the potential of the inner sphere:

$$\phi(r_1) = -\int_{\infty}^{r_3} \frac{Q}{4\pi\epsilon_0 r^2}\,dr + \left\{-\int_{r_2}^{r_1} \frac{Q}{4\pi\epsilon_0 r^2}\,dr\right.$$

Q. A static electric charge is distributed in a spherical shell of inner radius R_1 and outer radius R_2. The electric charge density is given by: $\rho = a + br$, where r is the distance from the center, and is zero everywhere else.

a) Find an expression for the electric field everywhere in terms of r.

b) Find expressions for the electric potential and energy density for $r < R_1$. Take the potential to be zero at $r \to \infty$.

Soln: Noting that ρ is a fn of only the radius r, we can take a concentric spherical surface of radius r as the Gaussian surface in accordance with the symmetry requirement.

Using Gauss's Law, we can get the following results:

a) Electric field strength.

For $r < R_1$, $E_1 = 0$

For $R_1 < r < R_2$, using the relation:

$$4\pi r^2 E_2 = \frac{4\pi}{\epsilon_0}\int_{R_1}^{r}(a + br')r'^2\,dr' \quad ; \text{ we find}$$

$$\vec{E}_2 = \frac{1}{\epsilon_0 r^3}\left[\frac{a}{3}(r^3 - R_1^3) + \frac{b}{4}(r^4 - R_1^4)\right]\vec{r}$$

For $R_2 > r$, from $4\pi r^2 E_3 = \frac{4\pi}{\epsilon_0}\int_{R_1}^{R_2}(a + br')r'^2 dr'$, we get:

$$\vec{E}_3 = \frac{1}{\epsilon_0 r^3}\left[\frac{a}{3}(R_2^3 - R_1^3) + \frac{b}{4}(R_2^4 - R_1^4)\right]\vec{r}$$

b) Potential and the energy density for $r < R_1$.

Noting that $\phi(\infty) = 0$, the potential is:

$$\phi(r) = -\int_\infty^r \vec{E}\cdot d\vec{r} = -\left(\int_\infty^{R_2} + \int_{R_2}^{R_1} + \int_{R_1}^{r}\right)\vec{E}\cdot d\vec{r}$$

$$= \frac{1}{\epsilon_0}\left[\frac{a}{3}(R_2^2 - R_1^2) + \frac{b}{4}(R_2^3 - R_1^3)\right]$$

Also, as $E_1 = 0$ $(r < R_1)$, the energy density for $r < R_1$ is

$$W = \frac{\epsilon_0}{2}E_1^2 = 0.$$

Q. An electric dipole consists of two equal charges q of opposite sign separated by a small distance a as shown in fig. (a) Show that the electric field at P is $\left[(1/4\pi\epsilon_0)\cdot\left(\frac{qa}{r_p^3}\right)\right]\hat{i}$ if $r_p \gg a$.

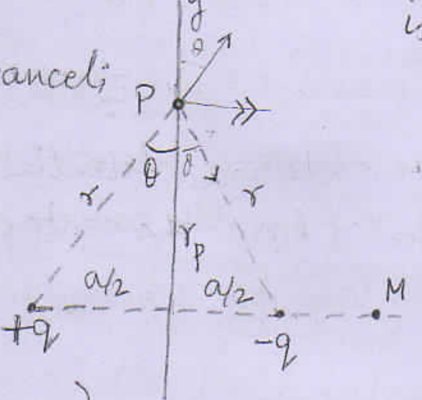
(b) Show that the field at M is $-\left[(1/2\pi\epsilon_0)(qa/r_m^3)\right]\hat{i}$ if $r_m \gg a$

Ans:

(a) At P, vertical components cancel;

Only an x component exists;

$E_p = \left[(2q)/(4\pi\epsilon_0 r^2)\right]\sin\theta$.



Substituting for $\sin\theta$ and r^2,

$$E_P = \left[\left(\frac{2q}{4\pi\epsilon_0}\right)\cdot\frac{a/2}{(r_p^2 + a^2/4)^{3/2}}\right].$$

$$r^2 = r_p^2 + \frac{a^2}{4}.$$

$$\sin\theta = \frac{a/2}{r} = \frac{a}{2\sqrt{r_p^2 + (a/2)^2}}$$

If $r_p \gg a/2$, then $a/2$ can be neglected in the denominator and hence we have:

$$\boxed{\vec{E_P} = \frac{qa}{4\pi\epsilon_0 r_p^3}\hat{i}}$$

(b) At point M the $-q$ charge is closer, which causes E_M to be in the $-X$ direction.

P.T.O.

$$E_M = \left[\frac{q}{4\pi\epsilon_0}\right]\cdot\left[\frac{1}{(r_m - a/2)^2} - \frac{1}{(r_m + a/2)^2}\right]$$

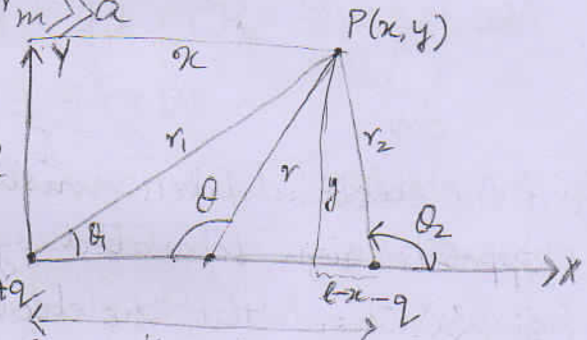
$$\Rightarrow E_M = \left[\frac{q}{4\pi\epsilon_0 r_m^2}\right]\left[\frac{1}{(1 - a/2r_m)^2} - \frac{1}{\left(1 + \frac{a}{2r_m}\right)^2}\right]$$

For $r_m \gg a/2 \Rightarrow \left(1 - a/2r_m\right)^{-2} \approx 1 + a/r_m$.

$$\left(1 + a/2r_m\right)^{-2} \approx 1 - a/r_m.$$

$$\therefore E_M = \left[\frac{q}{4\pi\epsilon_0 r_m^2}\right]\left[\left(\cancel{1} + a/r_m\right) - \left(\cancel{1} - a/r_m\right)\right]$$

$$\Rightarrow \boxed{\vec{E}_M = \frac{-2qa}{4\pi\epsilon_0 r_m^3}\,\hat{i}} \quad \text{for } r_m \gg a$$



Q. Write an expression for the potential $\phi(x,y)$ at a general point $P(x,y)$ in the XY plane.

Since $r_1 = (x^2+y^2)^{1/2}$ and $r_2 = [(l-x)^2+y^2]^{1/2}$

$$\phi(x,y) = \frac{1}{4\pi\epsilon_0}\cdot\left(\frac{q}{r_1} + \frac{(-q)}{r_2}\right) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{(x^2+y^2)^{1/2}} + \frac{(-q)}{(x^2+y^2+l^2-2xl)^{1/2}}\right]$$

Dipole moment: $\mu \equiv |q|l$.

Find the electric potential of the dipole at a point P whose distance r from the center of the dipole is large compared to l.

The law of cosines gives: $\boxed{\begin{aligned} r_1^2 &= r^2 + (l/2)^2 - rl\cos\theta \\ r_2^2 &= r^2 + (l/2)^2 + rl\cos\theta \end{aligned}}$

so that $r_2^2 - r_1^2 = 2rl\cos\theta$.

$$\therefore \phi = \frac{|q|}{4\pi\epsilon_0}\cdot\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{|q|}{4\pi\epsilon_0}\cdot\frac{r_2^2 - r_1^2}{r_1 r_2 (r_2 + r_1)}$$

$$\Rightarrow \phi = \frac{|q|}{4\pi\epsilon_0}\cdot\frac{2rl\cos\theta}{r_2 r_1 (r_2 + r_1)}$$

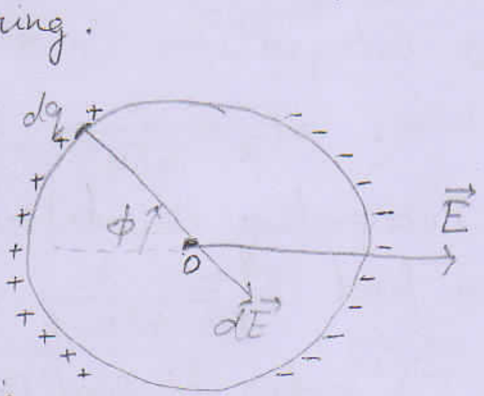
But, when $r \gg l$, we have $r_2 r_1 \approx r^2$ and $r_2 + r_1 \approx 2r$, giving

$$\phi \approx \frac{|q|}{4\pi\epsilon_0}\cdot\frac{2rl\cos\theta}{2r^3} = \frac{\mu\cos\theta}{4\pi\epsilon_0 r^2}.$$

<u>Note</u>: This approximate formula becomes exact when $l \to 0$ and $|q| \to \infty$ such that μ remains constant and the two charges form a Point Dipole

Q. A thin nonconducting ring of radius R is charged with a linear density $\lambda = \lambda_0 \cos\phi$, where λ_0 is a positive constant and ϕ is the azimuth angle. Find the electric field intensity E at the centre of the ring.

Ans. The given charge distribution is shown in Fig. The symmetry of this distribution implies that vector $\vec{E}$ at the point O is directed to the right, and its magnitude is equal to the sum of the projections onto the direction of $\vec{E}$ of vectors $d\vec{E}$ from elementary charges dq. The projection of vector $d\vec{E}$ onto vector $\vec{E}$ is:



$$dE\cos\phi = \frac{1}{4\pi\epsilon_0}\cdot\frac{dq}{R^2}\cos\phi \,. \longrightarrow (1)$$

where $dq = \lambda R\,d\phi = \lambda_0 R\cos\phi\,d\phi$.

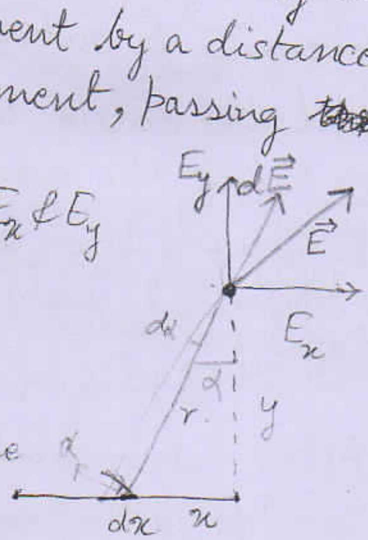
Integrating (1) over ϕ between 0 and 2π, we find the magnitude of the vector $\vec{E}$:

$$E = \frac{\lambda_0}{4\pi\epsilon_0 R}\int_0^{2\pi}\cos^2\phi\,d\phi = \frac{\lambda_0}{4\epsilon_0 R}.$$

* It should be noted that this integral is evaluated in the most simple way if we take into account that $\langle\cos^2\phi\rangle = 1/2$. Then $\int_0^{2\pi}\cos^2\phi\,d\phi = \langle\cos^2\phi\rangle\cdot 2\pi = \pi$.

Q. A semi-infinite ~~plane~~ straight uniformly charged ~~filament~~ filament has a charge λ per unit length. Find the magnitude and the direction of the field intensity at the point separated from the filament by a distance y and lying on the normal to the filament, passing ~~through~~ through its end.

Ans.: The problem is reduced to finding E_x & E_y viz. the projections of vector $\vec{E}$ (where it is assumed that $\lambda > 0$).



Let us start with E_x.

The contribution to E_x from the charge element of the segment dx is:

$$dE_x = \frac{1}{4\pi\epsilon_0}\cdot\frac{\lambda\,dx}{r^2}\sin\alpha \;\cdot \longrightarrow (1)$$

Let us reduce this expression to the form convenient for integration. In our case, $dx = r\,d\alpha/\cos\alpha$, $r = y/\cos\alpha$.
Then, $dE_x = \frac{\lambda}{4\pi\epsilon_0 y}\cdot\sin\alpha\, d\alpha$.

Integrating this expression ~~of~~ over α between 0 and $\pi/2$, we find $E_x = \frac{\lambda}{4\pi\epsilon_0 y}$.

In order to find the projection E_y, it is sufficient to recall that dE_y differs from dE_x in the $\sin\alpha$ in (1) is simply replaced by $\cos\alpha$.

This gives : $dE_y = (\lambda\cos\alpha\, d\alpha)/4\pi\epsilon_0 y$ and $E_y = \frac{\lambda}{4\pi\epsilon_0 y}$.

We have obtained an interesting result: $E_x = E_y$ ~~[illegible]~~ independently of y (i.e. vector $\vec{E}$ is oriented at the angle of 45° to the filament.) The modulus of vector $\vec{E}$ is,

$$E = \sqrt{E_x^2 + E_y^2} = (\lambda\sqrt{2})/4\pi\epsilon_0 y\,.$$

~~Problem on Gauss's Law~~

Q. A system consists of a uniformly charged sphere of radius R and a surrounding medium filled by a charge with the volume density $\rho = \alpha/r$, where α is a positive constant and r is the distance from the centre of the sphere. Find the charge of the sphere for which the electric field intensity E outsides the sphere is independent of r. Find the value of E.

Ans.: Let the sought charge of the sphere be q. Then, using the Gauss's law, we can write the following expression for a spherical surface of radius r (outside the sphere with charge q): $E\cdot 4\pi r^2 = \frac{q}{\epsilon_0} + \frac{1}{\epsilon_0}\int_R^r \frac{\alpha}{r}\,4\pi r^2\,dr$

After integration, we transform this eqn. to:

$$E\cdot 4\pi r^2 = (q - 2\pi\alpha R^2)/\epsilon_0 + 4\pi\alpha r^2/2\epsilon_0\,.$$

Electric Dipole: 3

$$V=\frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+}-\frac{q}{r_-}\right)=\frac{1}{4\pi\epsilon_0}\,\frac{q(r_- - r_+)}{r_+ r_-}$$

Since $r \gg l$,

From fig. 1 we see

$$r_- - r_+ = l\cos\theta.$$

& $r_+ r_- = r^2$,

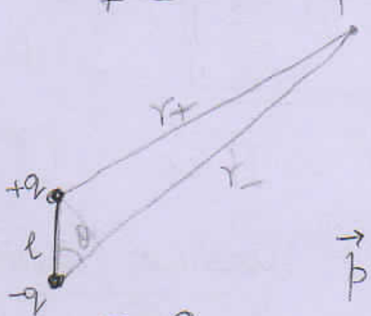


Fig 1

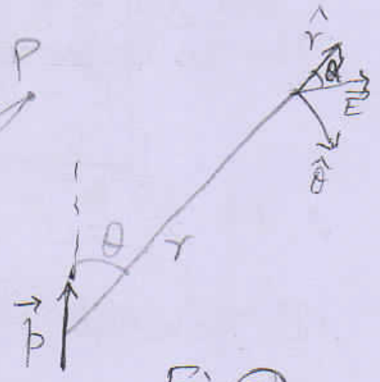


Fig 2

where r is the distance of point P to the dipole.

$$\boxed{V=\frac{1}{4\pi\epsilon_0}\,\frac{p\cos\theta}{r^2}}\ ; \text{ where } \vec{p}=q\vec{l}. \text{ (Dipole moment of the dipole.)}$$

$(r>l,\ \vec{l}\parallel\vec{p})$ $\vec{p}\rightarrow$ From '−ve' to +ve charge

Dipole Field:

$$E_r=-\frac{\partial V}{\partial r}=\frac{1}{4\pi\epsilon_0}\cdot\frac{2p\cos\theta}{r^3}$$

$$E_\theta=-\frac{1}{r}\cdot\frac{\partial V}{\partial\theta}=\frac{1}{4\pi\epsilon_0}\cdot\frac{p\sin\theta}{r^3}$$

∴ The modulus of vector $\vec{E}$ will be:

$$\boxed{E=\sqrt{E_r^2+E_\theta^2}=\frac{1}{4\pi\epsilon_0}\cdot\frac{p}{r^3}\sqrt{1+3\cos^2\theta}}.$$

In particular, for $\theta=0$ and $\theta=\pi/2$, we obtain the expressions for the field intensity on the dipole axis $(E_{\parallel})$ and the normal to it $(E_{\perp})$.

$$\boxed{E_{\parallel}=\frac{1}{4\pi\epsilon_0}\cdot\frac{2p}{r^3}}\ \&\ \boxed{E_{\perp}=\frac{1}{4\pi\epsilon_0}\cdot\frac{p}{r^3}}.$$

i.e. for the same r, the intensity $E_{\parallel}$ is twice as high as $E_{\perp}$.

Force acting on a dipole:

Let us place a dipole into a non-uniform electric field. Suppose that E_+ and E_- are the intensities of the external field at the points where the +ve and −ve dipole charges are located. Then the resultant force $\vec{F}$ acting on the dipole is:

$$\vec{F}=q\vec{E}_+-q\vec{E}_-=q(\vec{E}_+-\vec{E}_-).$$

The difference $\vec{E}_+-\vec{E}_-$ is the increment $\Delta\vec{E}$ of vector $\vec{E}$ on the segment equal to the dipole length l in the direction of vector $\vec{l}$. Since the length of the ~~vector~~ segment is small, we can write:

$$\Delta\vec{E} = \vec{E}_+ - \vec{E}_- = \frac{\Delta\vec{E}}{\Delta l}\,l = \frac{\partial\vec{E}}{\partial l}\,l$$

$$\therefore \boxed{\vec{F} = p\,\frac{\partial\vec{E}}{\partial l}} \qquad p = ql,\ \frac{\partial\vec{E}}{\partial l} \rightarrow \text{directional derivative of the vector}$$

- If $\frac{\partial\vec{E}}{\partial l} = 0 \Rightarrow \vec{F} = 0$. (For a uniform field)
- In general $\vec{F}$ does not coincide with ~~vector~~ $\vec{E}$ or $\vec{p}$.
- $\vec{F}$ coincides in direction only with the increment of $\vec{E}$, taken along the direction of $\vec{l}$ or $\vec{p}$.

Moment of forces acting on a Dipole:

$F_+ = qE_+$ & $F_- = -qE_-$ (w.r.t. centre of mass C)

$$\therefore \vec{M} = [\vec{r}_+ \times \vec{F}_+] + [\vec{r}_- \times \vec{F}_-] = [\vec{r}_+ \times q\vec{E}_+] - [\vec{r}_- \times q\vec{E}_-]$$

where $\vec{r}_+$ and $\vec{r}_-$ are the radius vectors of the charges $+q$ and $-q$ relative to the point C. For sufficiently small dipole length, $\vec{E}_+ \approx \vec{E}_-$ and $\vec{M} = [(\vec{r}_+ - \vec{r}_-) \times q\vec{E}]$

$$\vec{r}_+ - \vec{r}_- = \vec{l} \text{ and } q\vec{l} = \vec{p} \Rightarrow \boxed{\vec{M} = [\vec{p} \times \vec{E}]}$$

This moment of force tends to rotate the dipole so that its electric moment $\vec{p}$ is oriented along the external field $\vec{E}$. Such a position of the dipole is stable.

The energy of a dipole in an external field:

$$W = q_+V_+ + q_-V_- = q(V_+ - V_-)$$

$$V_+ - V_- \cong \frac{\partial V}{\partial l}\,l = -E_l \cdot l = -\vec{E}\cdot\vec{l}$$

$V_\pm \rightarrow$ Potentials of the external field at $+q$ & $-q$ charges' locations.

$$\therefore W = q\{-(\vec{E}\cdot\vec{l})\} \Rightarrow \boxed{W = -\vec{p}\cdot\vec{E}}$$

W is W_{min}, when $\vec{p} \uparrow\uparrow \vec{E}$. (Position of stable equilibrium)

- If it is displaced from this position, the moment of external forces will return the dipole to the equilibrium position.