Magnetic vector potential

From Gauss' theorem in magnetism $\nabla \cdot \vec{B} = 0$

But $\nabla \cdot (\nabla \times \vec{A}) = 0$ $\therefore \boxed{\vec{B} = \nabla \times \vec{A}}$

$\vec{A}$ is called magnetic vector potential.

We know that $\nabla \cdot \vec{B} = 0$; but $\vec{B} = -\mu_0 \nabla \phi \Rightarrow \nabla^2 \phi = 0$ which is Laplace's equation.

In case where the area of magnetic field concerned is threaded by currents, then $\nabla \times \vec{B}$ has a finite value ; the field is now rotational but still solenoidal.

Since $\nabla \times \vec{B} \neq 0$ in this case, the idea of magnetostatic scalar potential ϕ no longer applies. [$\vec{B} = -\mu_0 \nabla \phi$; $\nabla \times \vec{B} = -\mu_0 \nabla \times (\nabla \phi$]

A new type of potential, the magnetic vector potential ($\vec{A}$) has to be introduced & it is given by $\vec{B} = \nabla \times \vec{A}$ $\longrightarrow (1)$

or, $\nabla \times \vec{B} = \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 A$ $\longrightarrow (2)$

Now eqn. (1) does not define $\vec{A}$ completely since $\nabla \times (A + \nabla u) = \nabla \times \vec{A}$ [$\because \nabla \times (\nabla u) = (\nabla \times \nabla) u = 0$] where u is any scalar function. To give a full definition (Yarwood p-747) $\nabla \cdot \vec{A}$ must also be zero.

Hence eqn. (2) becomes $\nabla \times \vec{B} = -\nabla^2 \vec{A}$

But from Ampere's circuital law $\nabla \times \vec{B} = \mu_0 \vec{J}$

$$\therefore \boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

This eqn. is of the form of Poisson's eqn., but where vector potentials ($\vec{A}$) instead of scalar potential (ϕ) are concerned.

Now $\vec{A} = \hat{i} A_x + \hat{j} A_y + \hat{k} A_z$ & $\vec{J} = \hat{i} J_x + \hat{j} J_y + \hat{k} J_z$

$$\therefore \nabla^2 (\hat{i} A_x + \hat{j} A_y + \hat{k} A_z) = -\mu_0 (\hat{i} J_x + \hat{j} J_y + \hat{k} J_z)$$

Equating the components, $\nabla^2 A_x = -\mu_0 J_x$

$\nabla^2 A_y = -\mu_0 J_y$, $\nabla^2 A_z = -\mu_0 J_z$

$\Rightarrow \nabla\cdot(\nabla A_x) = -\mu_0 J_x$

$$\int \nabla\cdot(\nabla A_x)\, dV = -\int \mu_0 J_x\, dV$$

or, $\oint (\nabla A_x)\cdot \vec{ds} = -\int \mu_0 J_x\, dV$

or $\frac{\partial A_x}{\partial r}\, 4\pi r^2 = -\int \mu_0 J_x\, dV$ $\qquad \nabla A_x = \frac{\partial A_x}{\partial r}$

or $\partial A_x = -\int \mu_0 J_x\, dV\, \frac{\partial r}{4\pi r^2}$ $\qquad \oint ds = 4\pi r^2$

Int. $A_x = -\int \mu_0 J_x\, dV \int \frac{\partial r}{4\pi r^2} = \int \frac{\mu_0 J_x\, dV}{4\pi r}$

Similarly $A_y = \int \frac{\mu_0 J_y\, dV}{4\pi r}$, $A_z = \int \frac{\mu_0 J_z\, dV}{4\pi r}$

Adding $\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\, dV}{r} = \frac{\mu_0}{4\pi}\int \frac{I\, \vec{dl}}{r}$ $\quad \left[\because \vec{J}\, dV = \vec{J}\, \vec{ds}\cdot\vec{dl} = I\, dl\right]$

Electric field is given by $\nabla\times\vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\vec{A})$

or, $\nabla\times\left(\vec{E} + \frac{\partial \vec{A}}{\partial t}\right) = 0$

$\therefore \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla\phi$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

Magnetic effect of steady current (Mag. vector potential)

C.U. 2010

What is magnetic vector potential? Using the concept of vector potential deduce Biot - Savart law.

Ans We know $\nabla \cdot \vec{B} = 0$ always. Again $\nabla \cdot (\nabla \times \vec{A}) = 0$ where $\vec{A}$ is a vector. Hence $\vec{B} = \nabla \times \vec{A}$.

$\vec{A}$ is called magnetic vector potential which is defined as a vector function the curl of which is equal to $\vec{B}$ (the magnetic induction field).

$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}\, dV}{r} = \frac{\mu_0}{4\pi} \int \frac{I\, \vec{dl}}{r}$$

Differentiating $$d\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{J}\, dV}{r} = \frac{\mu_0}{4\pi} \frac{I\, \vec{dl}}{r}$$

[$\vec{dl} \rightarrow$ length of the path having an area through which I is flowing & $\vec{r}$ is the vector drawn from $\vec{dl}$ to the point P where the value of $\vec{A}$ is to be determined]

Now $\vec{B} = \nabla \times \vec{A}$

$$d\vec{B} = \nabla \times d\vec{A}$$

$$= \nabla \times \frac{\mu_0}{4\pi} \frac{I \vec{dl}}{r}$$

$$= \frac{\mu_0 I}{4\pi} (\nabla \times \vec{dl}) \frac{1}{r}$$

$$= -\frac{\mu_0 I}{4\pi} (\vec{dl} \times \vec{\nabla}) \frac{1}{r} = -\frac{\mu_0 I}{4\pi} \vec{dl} \times \vec{\nabla} \frac{1}{r}$$

$$= -\frac{\mu_0 I}{4\pi} \vec{dl} \times \left(-\frac{\vec{r}}{r^3}\right) = \frac{\mu_0 I}{4\pi} \frac{\vec{dl} \times \vec{r}}{r^3}$$

or $$\boxed{d\vec{B} = \frac{\mu_0}{4\pi} \frac{I \vec{dl} \times \vec{r}}{r^3}} \longrightarrow \text{Biot - Savart law.}$$

Ampere's law from mag. vector potential

$$\oint_C \vec{B} \cdot \vec{dl} = \oint_C (\nabla \times \vec{A}) \cdot \vec{dl} = \int_S \nabla \times (\nabla \times \vec{A}) \cdot \vec{dS}$$

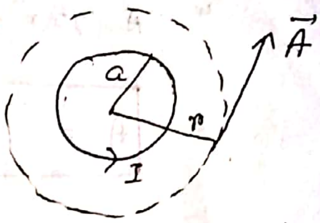
Now $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$ [we choose $\because \nabla \cdot \vec{A} = 0$]

$$\therefore \oint \vec{B} \cdot \vec{dl} = \int_S \mu_0 \vec{J} \cdot \vec{dS} = \mu_0 I,$$ which is Ampere's law.

Problems

① Determine magnetic vector potential at a distance r from the axis of an infinitely long solenoid of radius a having n number of turns per unit length & carrying current I.

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{I\,\vec{dl}}{r}$$



i.e., $\vec{A}$ is in the direction of current I.

Now I flows along the circumference of the solenoid, i.e., $\vec{A}$ is $\perp^r$ to $\vec{r}$.

Now, $\oint \vec{A}\cdot\vec{dl} = \int_s (\nabla\times\vec{A})\cdot\vec{ds} = \int_s \vec{B}\cdot\vec{ds}$

or $\quad 2\pi r A = (\mu_0 n I)\pi a^2$

or $\quad \boxed{A = \frac{\mu_0 n I a^2}{2r}}$ (for $r > a$)

i.e., $\vec{A}$ is not zero outside the solenoid though $\vec{B} = 0$ there.

For $r < a$ (inside the solenoid)

$$2\pi r A = (\mu_0 n I)\pi r^2$$

or $\quad \boxed{A = \frac{\mu_0 n I r}{2}}$ (for $r < a$)

② Find $\vec{B}$ & $\vec{A}$ for a solid cylindrical conductor of radius a, where the current I is uniformly distributed over the cross-section,

$\oint \vec{B}\cdot\vec{dl} = \mu_0 I_{enc}$. [I_{enc} is the current enclosed by the path of radius r, i.e.,

$I_{enc} = \frac{I}{\pi a^2}\cdot\pi r^2 = \frac{I r^2}{a^2}$

or $\quad B\cdot 2\pi r = \mu_0 I \frac{r^2}{a^2}$

or $\quad B = \frac{\mu_0 I r}{2\pi a^2}$ [for $r < a$]

For $r > a$ $\quad B\cdot 2\pi r = \mu_0 I$

$$B = \frac{\mu_0 I}{2\pi r}$$

To find vector potential $\vec{A}$ $(r<a)$

$$\oint \vec{A}\cdot\vec{dl} = \int_s \vec{B}\cdot\vec{ds} = \frac{\mu_0 I r}{2\pi a^2}\cdot\pi r^2$$

or $A\cdot 2\pi r = \dfrac{\mu_0 I r}{2\pi a^2}\,\pi r^2$

or $\boxed{A = \dfrac{\mu_0 I}{4\pi a^2}\,r^2}$ (for $r<a$)

For $r>a$

$$A\cdot 2\pi r = \frac{\mu_0 I}{2\pi r}\cdot\pi a^2$$

or $\boxed{A = \dfrac{\mu_0 I}{4\pi r^2}\,a^2}$

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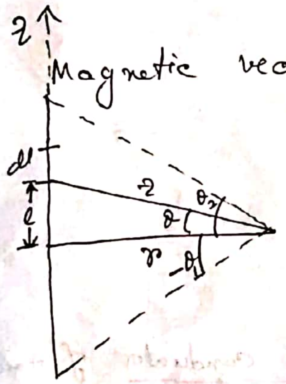
(3) Determine the magnetic vector potential at a distance r from a very long thin straight wire carrying a current I. Hence find the corresponding negative field $\vec{B}$. (5)

Ans Magnetic vector potential $\vec{A} = \dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I\,\vec{dl}}{r}$



Here $\vec{A} = \hat{z}\dfrac{\mu_0}{4\pi}\displaystyle\int \frac{I\,dl}{z}$ $\left[\because \vec{dl} = \hat{z}\,dl\right]$

Now, $l = r\tan\theta \Rightarrow dl = r\sec^2\theta\, d\theta$

$z = r\sec\theta$

$$\therefore A = \frac{\mu_0 I}{4\pi}\int_{-\theta_1}^{\theta_2} \frac{r\sec^2\theta\, d\theta}{r\sec\theta} = \frac{\mu_0 I}{4\pi}\int_{-\theta_1}^{\theta_2}\sec\theta\, d\theta$$

$$= 2\cdot\frac{\mu_0 I}{4\pi}\int_0^{\theta_1}\sec\theta\, d\theta \quad \left[\because L >> r \rightarrow \theta_1 = \theta_2\right]$$

$$= \frac{\mu_0 I}{2\pi}\ln\left(\sec\theta_1 + \tan\theta_1\right)$$

$\sec\theta_1 = \dfrac{z}{r}$ & $\tan\theta_1 = \dfrac{L/2}{r}$; as $L >> r$, $L/2 \simeq z$ ($L \rightarrow$ total length)

$\therefore \sec\theta_1 \simeq \tan\theta_1 = \dfrac{L}{2r}$

$$\therefore \boxed{\vec{A} = \frac{\mu_0 I}{2\pi}\ln\left(\frac{L}{r}\right)\hat{a}_z}$$

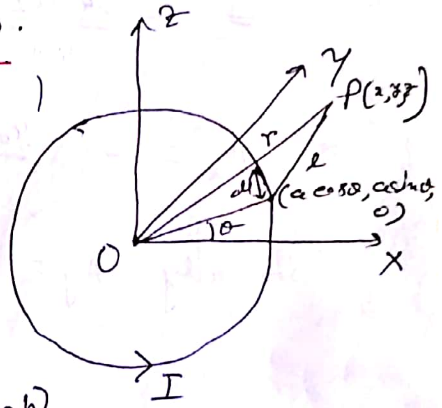
$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \frac{\mu_0 I}{2\pi}\ln\frac{L}{r} \end{vmatrix}$$

$$= \hat{r}\,\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{\mu_0 I}{2\pi}\ln\frac{L}{r}\right) - \hat{\theta}\,\frac{\partial}{\partial r}\left(\frac{\mu_0 I}{2\pi}\ln\frac{L}{r}\right) + 0$$

$$= \hat{\theta}\,\frac{\mu_0 I}{2\pi}\cdot\frac{1}{r}$$

$$\therefore \boxed{\vec{B} = \frac{\mu_0 I}{2\pi r}\,\hat{\theta}}$$

(4) Find the magnetic vector potential and hence magnetic field due to a small current loop.



Let the current loop is in XY-plane & Z-axis is perpendicular to it. We consider an element of the loop of length dl between θ & $\theta + d\theta$.

$\therefore\ dl = a\,d\theta$ ($a \rightarrow$ radius of the loop)

The components of dl along coordinate axes are

$dl_x = d(a\cos\theta) = -a\sin\theta\, d\theta$

$dl_y = d(a\sin\theta) = a\cos\theta\, d\theta$; $dl_z = 0$

The magnetic vector potential at P due to the loop is given by $\vec{A} = \frac{\mu_0 I}{4\pi}\int \frac{\vec{dl}}{l}$

Its x component is $A_x = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{-a\sin\theta\, d\theta}{l}$

Now from the geometry of the fig.,

$$l^2 = (x - a\cos\theta)^2 + (y - a\sin\theta)^2 + z^2$$

$$= x^2 + y^2 + z^2 - 2ax\cos\theta - 2ay\sin\theta + a^2(\cos^2\theta + \sin^2\theta)$$

$$= r^2 - 2ax\cos\theta - 2ay\sin\theta \quad [\because a \ll r]$$

or, $l^2 = r^2\left(1 - \frac{2ax\cos\theta}{r^2} - \frac{2ay\sin\theta}{r^2}\right)$

$$\Rightarrow \frac{1}{l} = \frac{1}{r}\left(1 - \frac{2ax\cos\theta}{r^2} - \frac{2ay\sin\theta}{r^2}\right)^{-1/2}$$

$$= \frac{1}{r}\left[1 + \frac{ax\cos\theta + ay\sin\theta}{r^2}\right]$$ neglecting higher order terms

$$\therefore A_x = -\frac{\mu_0 a I}{4\pi}\int_0^{2\pi} \frac{1}{r}\left(1 + \frac{ax\cos\theta + ay\sin\theta}{r^2}\right)\sin\theta\, d\theta$$

$$= -\frac{\mu_0 a I}{4\pi}\left[\frac{1}{r}\underbrace{\int_0^{2\pi}\sin\theta\, d\theta}_{0} + \frac{ax}{r^3}\underbrace{\int_0^{2\pi}\sin\theta\cos\theta\, d\theta}_{0} + \frac{ay}{r^3}\int_0^{2\pi}\sin^2\theta\, d\theta\right]$$

$$= -\frac{\mu_0 a I}{4\pi}\frac{ay}{r^3}\cdot\pi$$

or $$A_x = -\frac{\mu_0 I(\pi a^2) y}{4\pi r^3}$$

Similarly $$A_y = \frac{\mu_0 I}{4\pi}\int \frac{dl_y}{l} = \frac{\mu_0 I}{4\pi}\int \frac{a\cos\theta\, d\theta}{l}$$

$$= \frac{\mu_0 I(\pi a^2) x}{4\pi r^3}$$

& $A_z = 0$

$$\therefore \vec{A} = \frac{\mu_0 I}{4\pi r^3}(\pi a^2)(-\hat{i}y + \hat{j}x) = \frac{\mu_0 I a^2}{4}\left(-\hat{i}\frac{y}{r^3} + \hat{j}\frac{x}{r^3}\right)$$

$$\therefore \vec{B} = \nabla\times\vec{A} = \frac{\mu_0 I a^2}{4}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\frac{y}{r^3} & \frac{x}{r^3} & 0 \end{vmatrix}$$

or $$\vec{B} = \frac{\mu_0 I a^2}{4}\left[-\hat{i}\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \hat{j}\frac{\partial}{\partial z}\left(\frac{y}{r^3}\right) + \hat{k}\left\{\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r^3}\right)\right\}\right]$$

$$= \frac{\mu_0 I a^2}{4}\left[\hat{i}\frac{3}{2}\frac{x\cdot 2z}{(x^2+y^2+z^2)^{5/2}} + \hat{j}\frac{3}{2}\frac{y\cdot 2z}{(x^2+y^2+z^2)^{5/2}} - \hat{k}\left\{\frac{3}{2}\frac{x\cdot 2x}{(x^2+y^2+z^2)^{5/2}} + \frac{3}{2}\frac{y\cdot 2y}{(x^2+y^2+z^2)^{5/2}} + \frac{1}{r^3} + \frac{1}{r^3}\right\}\right]$$

On the axis of the loop $x = y = 0$

$$\therefore \vec{B} = \frac{\mu_0 I a^2}{4} \cdot \frac{2}{r^3} = \frac{\mu_0 I a^2}{2(z^2+a^2)^{3/2}} \quad \left[\because r^2 = x^2+y^2+z^2 \\ = a^2+z^2\right]$$